SCHUR MULTIPLIERS OF NILPOTENT LIE ALGEBRAS

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Abstract. We consider the Schur multipliers of finite dimensional nilpotent Lie algebras. If the algebra has dimension greater than one, then the Schur multiplier is non-zero. We give a direct proof of an upper bound for the dimension of the Schur multiplier as a function of class and the minimum number of generators of the algebra. We then compare this bound with another known bound.

Keywords: Schur multiplier

1. Introduction

The Lie algebra analogue of the Schur multiplier was investigated in the dissertations of Kay Moneyhun and Peggy Batten (see [8] and [2]). Among their results is that if $\dim L = n$, then $\dim M(L) \leq \frac{1}{2}n(n-1)$ where $M(L)$ is the Schur multiplier of the Lie algebra, L. A number of other results bounding $\dim M(L)$ have appeared (see [3], [5], [9], [10], and [13]). In this note several other bounds are provided. In particular, we show that if L is nilpotent and $\dim L > 1$, then $\dim M(L) \neq 0$. This result can be contrasted with a result of Johnson in [6] which shows that a $p-$group with trivial multiplier has restrictions placed on it. We also find an upper bound for $\dim M(L)$ as a function of class and the number of generators for L. This result is similar to Theorem 3.2.5 of [7]. This Lie algebra result follows as a consequence of a general theory, provided in [10] but we give a short, direct proof and contrast this bound with one found in [5].

2. Preliminaries

Suppose that L is generated by n elements. Let F be a free Lie algebra generated by n elements and $L \cong F/R$. Since R is an ideal in

F, R is also free. Witt's formula from [1] gives us

$$\dim F^d/F^{d+1} = \frac{1}{d}\sum_{m|d} \mu(m)n^{d/m} \equiv l_n(d) \tag{2.1}$$

where μ is the Möbius function. Hence, F/F^t is finite dimensional and nilpotent for all t.

Let N be an ideal in L and S be an ideal in F such that $(S+R)/R \cong N$. Recall that $M(L) = (F^2 \cap R)/[F,R]$ ([2]). Then $M(L/N) \cong (F^2 \cap (S+R))/[F, S+R]$. It is routine to verify that there is a natural exact sequence

$$0 \to \frac{R \cap [F,S]}{[F,R] \cap [F,S]} \to M(L) \to M(L/N) \to \frac{N \cap L^2}{[N,L]} \to 0. \tag{2.2}$$

Note that covers and multipliers can be computed using the GAP program [4].

3. Trivial M(L)

It is shown in [6] that if G is a finite $p-$group with $M(G) = e$, then severe restrictions are placed on G. For further work in the problem see [11] and also [12] for a simpler proof. We will show that if L is a nilpotent Lie algebra with $M(L) = 0$ then $\dim L \leq 1$.

Let L be a nilpotent Lie algebra generated by $n > 1$ elements. Hence, $\dim L/L^2 = n$. Let F be a free Lie algebra generated by n elements with $L \cong F/R$. Suppose that L has class c. Hence, $F^{c+2} \subsetneq F^{c+1} \subseteq R$ using the result in the last section. Furthermore, F/F^{c+2} is finite dimensional and nilpotent of class $c+1$. Then,

$$n = \dim L/L^2 = \dim \frac{F/R}{(F/R)^2} = \dim \frac{F}{F^2+R} \leq \dim F/F^2 = n.$$

Lemma 3.1. *If L is nilpotent and $\dim L = n > 1$, then $R \subseteq F^2$ and $M(L) \cong R/[F,R]$.*

Theorem 3.2. *If L is a finite dimensional nilpotent Lie algebra of dimension greater than 1 and class c, then $M(L) \neq 0$.*

Proof. Continuing with the notation, $L \cong F/R$, $F^{c+2} \subsetneq F^{c+1} \subseteq R$, and F/F^{c+2} is nilpotent. Hence, $[F,R] \subsetneq R$ and $M(L) \cong R/[F,R] \neq 0$. □

4. An Upper Bound for $\dim M(L)$

Suppose L has class $c \geq 2$. Let $N = L^c$ and $S = F^c$ in Eq. 2.2. Then $L^c \cong (F^c+R)/R$ and $[F,S] = F^{c+1} \subseteq R$ since $L^{c+1} = 0$. Hence, Eq. 2.2 becomes

$$0 \to \frac{F^{c+1}}{[F,R]\cap F^{c+1}} \xrightarrow{\sigma} M(L) \to M(L/L^c) \to L^c \to 0. \tag{4.1}$$

Theorem 4.1. *Let L be a nilpotent Lie algebra of class c which is generated by n elements. Then*

$$\dim M(L) \leq \sum_{j=1}^{c} l_n(j+1)$$

where $l_n(q) = \frac{1}{q}\sum_{s|q}\mu(s)n^{q/s}$.

Proof. Induct on c. Let F be free of rank n where $L \cong F/R$. If $c = 1$, then F/R is abelian, $F^2 \subseteq R$ and $M(L) = F^2/[F,R]$. Since $F^3 \subseteq [F,R]$, $\dim M(L) \leq \dim F^2/F^3 = l_n(2)$. Now, suppose that $c > 1$. By induction, $\dim M(L/L^c) \leq t \equiv \sum_{j=1}^{c-1} l_n(j+1)$. In Eq. 4.1, let $A = Im(\sigma)$. Then $M(L)/A \cong B \subseteq M(L/L^c)$. Hence, $\dim M(L)/A \leq t$. But $F^{c+1} \subseteq R$ and $F^{c+2} \subseteq [F,R]\cap F^{c+1}$. Thus, A is the homomorphic image of F^{c+1}/F^{c+2}. Therefore $\dim A \leq l_n(c+1)$ by Eq. 2.1 and $\dim M(L) \leq t + l_n(c+1)$ as desired. □

We compare our result to the upper bound given in [5]:

Theorem 4.2. *If L is a Lie algebra of dimension n, then*

$$\dim M(L) \leq \frac{1}{2}n(n-1) - \dim L^2.$$

We now examine the two theorems applied to different Lie algebras.

Example 4.3. Let F be a free Lie algebra on 2 generators and $L = F/F^3$. Then L is a Lie algebra of 2 generators and class 2. So, $L \supseteq L^2 \supseteq L^3 = 0$. Then, $\dim L/L^2 = l_2(1) = 2$, $\dim L^2/L^3 = l_2(2) = \frac{1}{2}[\mu(1)2^2 + \mu(2)2] = \frac{1}{2}(4-2) = 1$. Thus, $\dim L = 3$ and by Theorem 4.2, $\dim M(L) \leq 2$. By Theorem 4.1,

$$\begin{aligned}\dim M(L) \leq \sum_{j=1}^{2} l_2(j+1) &= l_2(2) + l_2(3) \\ &= 1 + \frac{1}{3}(\mu(1)2^3 + \mu(3)2) \\ &= 1 + \frac{1}{3}(6) = 3.\end{aligned}$$

Thus, the result of Theorem 4.2 proves to be a better bound than the one obtained by our new theorem.

Example 4.4. Let F be a free Lie algebra of 2 generators and $L = F/F^4$. Then L is a Lie algebra of 2 generators and class 3. Thus, $\dim L = 5$ and by Hardy's Theorem, $\dim M(L) \leq 7$. By LB's Theorem,

$$\begin{aligned}\dim M(L) \leq \sum_{j=1}^{3} l_2(j+1) &= l_2(2) + l_2(3) + l_2(4) \\ &= 1 + 2 + \frac{1}{4}[\mu(1)2^4 + \mu(2)2^2 + \mu(4)2] \\ &= 3 + \frac{1}{4}(16 - 4) \\ &= 3 + 3 = 6\end{aligned}$$

Thus, we see that in this case, our theorem creates a better upper bound for $\dim M(L)$ than the previously known result.

References

[1] Y. Bahturin, Identical Relations in Lie algebras, V.N.U. Science Press BV, (1987).

[2] P. Batten and E. Stitzinger, On Covers of Lie Algebras, *Comm. in Alg*, 24 (1996).

[3] L. Bosko, On Schur Multipliers of Lie Algebra and Groups of Maximal Class, *Int. J. Of Algebra and Computations*, to appear.

[4] G. Ellis, HAP-Homological Algebra Programming A package for the computer algebra system GAP. http://www.gap-system.org/hap.html

[5] P. Hardy and E. Stitzinger, On Characterizing Nilpotent Lie Algebras by Their Multipliers, $t(L) = 3, 4, 5, 6$. *Comm. in Alg.* 26 (1998) 3527-3539.

[6] D. Johnson, A Property of $p-$Groups with Trivial Multiplicator. *Amer J. Math* 98, (1976) 105-108.

[7] G. Karpilovsky, The Schur Multiplier, Oxford Univ. Press, New York (1987).

[8] K. Moneyhun, Isoclinisms in Lie Algebras, *Alg. Groups and Geometries 11* (1994) 9-22.

[9] P. Niroomand and F. Russo, A Note on the Schur Multiplier of a Nilpotent Lie Algebra, *Comm. in Alg.*, to appear.

[10] A.R. Salemkar, B. Edalatzadeh, and M. Araskhan, Some Inequalities for the Dimension of the c-Nilpotent Multiplier of Lie Algebras, *J. of Algebra*, 322 (2009).1575-1585.

[11] V. Webb, The Schur Multiplier of a $p-$Group, *Trans. Amer. Math. Soc.* 291 (1985) 755-763.

[12] J. Wiegold, On a Result of Johnson about Schur Multipliers, *Glasgow Math. J.* 24 (1992) 347.

[13] B. Yankosky, On the Multiplier of a Lie Algebra, *J. of Lie Theory* 13 (2003) 1-6.