ELEMENTARY APPLICATIONS OF YOUNG DIAGRAMS AND YOUNG TABLEAUX

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Chapter 1

Young Diagrams and Graph Theory

1.1 Partitions and Young Diagrams

Definition 1.1. *A partition of n is a sequence of positive integers*

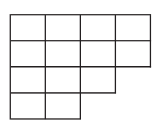
$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$$

where $\lambda_i \geq \lambda_{i+1}$ for $i = 1, 2, \ldots, k-1$ and $\sum_{i=1}^{k} \lambda_i = n$, denoted $\lambda \vdash n$. k is called the length of the partition and is denoted by $L(\lambda) = k$.

Example 1.1. *Two partitions of 13 are $\lambda = [4, 4, 3, 2]$ and $\sigma = [6, 2, 2, 2, 1]$, where $L(\lambda) = 4$ and $L(\sigma) = 5$.*

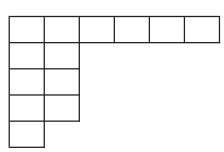
Definition 1.2. *A Young diagram or a Ferrers diagram of λ of length k is a collection of empty cells having k left-justified rows with row i containing λ_i cells for $1 \leq i \leq k$.*

Example 1.2. *For $\lambda = [4, 4, 3, 2] \vdash 13$, the corresponding Young diagram is*



This will be denoted as $Y(\lambda)$. Notice that the total number of cells is 13.

Example 1.3. *For $\sigma = [6, 2, 2, 2, 1] \vdash 13$, the corresponding Young diagram is*



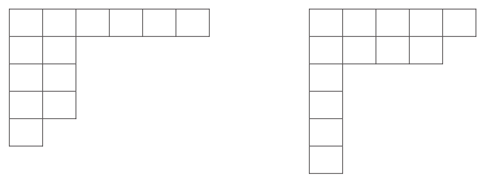
This will be denoted as $Y(\sigma)$, again the total number of cells is 13.

Observe from examples 1.2 and 1.3, there is a one to one correspondence between n and the number of cells in the Young diagram. We can also look at the conjugate diagram by flipping the diagram corresponding to λ over its main diagonal, from upper left to lower right. Suppose $\lambda \vdash n$. Then the **conjugate** of λ is the partition λ^* whose j^{th} part is the number of cells in column j of $Y(\lambda)$. In other words, the j^{th} part of λ^* is

$$\lambda_j^* = |(\{i \mid \lambda_i \geq j\})| \tag{1.1}$$

where the cardinality of a set, S, is denoted as $|S|$. We can also think of $Y(\lambda^*)$ as the transpose of $Y(\lambda)$.

Example 1.4. *The partition* $\lambda^* = [5, 4, 1, 1, 1, 1]$ *is the conjugate of the partition* $\lambda = [6, 2, 2, 2, 1]$. *This can be seen by looking each Young diagram* $Y(\lambda)$ *and* $Y(\lambda^*)$ *below.*



$Y(\lambda)$ $Y(\lambda^*)$

The length of λ^* is the $max_i\{\lambda_i\}$, for $1 \leq i \leq k$. Thus, $L(\lambda^*) = \lambda_1$. The partition λ is said to be **self conjugate** if $\lambda = \lambda^*$. Also observe that $Y(\lambda)$ is symmetric when $\lambda = \lambda^*$.

A **Young tableau** of a partition λ of n is a Young diagram of shape λ with integers $1, 2, \ldots, n$ bijectively filled in the cells. We denote T^λ as a Young tableau of shape λ.

Example 1.5. *For* $\lambda = [4, 4, 3, 2] \vdash 13$, *examples of Young tableaux of shape* λ *are:*

$T^\lambda =$

1	2	3	4
5	6	7	8
9	10	11	
12	13		

$S^\lambda =$

1	7	8	12
10	6	2	3
4	5	11	
9	13		

For each λ, we can see that the total number of tableaux of shape λ is $n!$.

Definition 1.3. *A Young tableau of shape* λ, T^λ, *is said to be standard if the integers* $1, 2, \ldots, n$ *are inserted in such a way that each row and each column are strictly increasing.*

Example 1.6. *For* $\lambda = [4, 4, 3, 2] \vdash 13$, *two examples of standard tableaux are:*

$T^\lambda =$

1	3	6	9
2	5	11	12
4	8	13	
7	10		

$S^\lambda =$

1	4	9	10
2	5	11	13
3	7	12	
6	8		

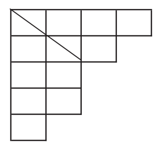
Notice that for each standard tableau, the number 1 will always be in the upper left cell. A useful calculation of a tableau is calculating the trace of partition λ.

Definition 1.4. *The trace of a partition λ is defined as*

$$tr(\lambda) = max\{i \mid \lambda_i \geq i\}.$$

The trace of a partition can be seen by its Young diagram; the trace of a partition λ is equal to the number of cells of the Young diagram of shape λ in the main diagonal.

Example 1.7. *For $\lambda = [4, 3, 2, 2, 1]$, we have that $\lambda_1 = 4 \geq 1, \lambda_2 = 3 \geq 2$, but $\lambda_3 = 2 \ngeq 1$. Thus, $tr(\lambda) = 2$. We can also look a Young diagram of shape λ that the main diagonal goes through 2 cells.*

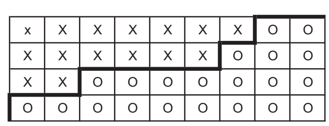


1.2 Lattice Paths

Frequently one refers to the relative positions in tableaux. To describe this, we will use compass directions such as: A cell a is North of cell b if it is in a row above cell b. A cell a is north of cell b if it is in the same row or a row above cell a. The same convention holds for the other directions as well as combinations of directions. For example, NorthEast describes all cells strictly above and to the right.

We will count the number of partitions with at most n parts and no part larger than m. The Young diagram for such a partition fits, left justified, in a rectangular grid with m rows and n columns and every left justified diagram inside the grid corresponds to a partition of the type that we are considering.

Example 1.8. *Let $n = 9$ and $m = 4$.*



is such a diagram and the partition is $\lambda = [7, 6, 2]$

We will defined the **vertices** of the grid, to be the top corners of each cell. All of the vertices of the cells in a grid are called a **lattice**. In the example there are $n + 1 = 10$ vertices in each row and $m + 1 = 5$ vertices in each column. Any sequence of horizontal and vertical moves between vertices is called a **lattice path**. We are interested in counting the number of lattice paths from the the most NorthEast vertex in the lattice to the most SouthWest vertex making only Westward horizontal moves and Southward vertical moves. How many such lattice paths exist? Each tableau outlines a path from the NorthEast vertex to the SouthWest vertex and these paths are examples of lattice paths. Clearly there is a one-to-one correspondence between Young diagrams and lattices paths which only use South and West moves.

Each of the lattice paths that we are considering has n moves West and m moves South. Label each move with a W or S and form a list, by putting them in order. In the example, the diagram determines the lattice path which has list WWSWSWWWWSWWS. The number of lists is the number of ways to label $m+n$ objects (the moves) with n W's and m E's. Therefore the number of possible lattice paths is

$$\begin{pmatrix} m+n \\ m \end{pmatrix} = \begin{pmatrix} m+n \\ n \end{pmatrix}$$

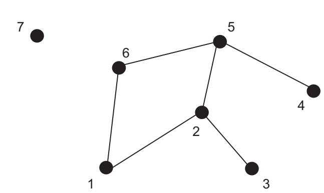
Exercises

1. How many partitions are there with exactly n parts and no part larger than m?
2. How many ways are there to go n blocks west and m blocks south as described in this section but that the first move must be West?
3. Let A be 10 blocks North and 6 blocks East of B. How many paths are there from A to B if the path must go through C which is 3 blocks North and 2 blocks East of B?
4. How many partitions are the that have 12 parts, no part larger than 5 and part 4 is 3?
5. Prove, using Young diagrams, that the number of partitions of n that have all parts odd is the same as the number of self conjugate partitions (they are equal to their conjugate)

1.3 Graph Theory

Definition 1.5. *A graph G is a set of vertices (nodes) v connected by edges (links) e. Thus G=(v,e).*

Example 1.9. *The graph on* $V = \{1, 2, 3, \ldots, 7\}$ *with* $E = \{(1, 2), (1, 6), (2, 3), (2, 5), (4, 5), (5, 6)\}$ *is shown below*



We denote $V(G)$ as the vertex set of a graph G and denote its edge set as $E(G)$. The number of vertices of a graph G is called the **order** of G and is denoted $|G|$. We refer to the graph of order 0 as the trivial graph. For this paper, we are interested in graphs that are **simple**, or a graph G where at most one edge may connect any two vertices, and non-directed, so that if $(a, b) \in E$ then $(b, a) \in E$. The graphs in the remainder of this paper will also be **connected**, or there must be a path from any vertex to any other vertex.

Definition 1.6. *The degree of a vertex V, denoted $d_G(V)$, is the number of edges connected to that vertex.*

Example 1.10. *From the figure in example 1.9, we have $|G| = 7$ and*

$$d_G(1) = 2, \;\; d_G(2) = 3, \;\; d_G(3) = 1 = d_G(4)$$
$$d_G(5) = 3, \;\; d_G(6) = 2, \;\; d_G(7) = 0$$

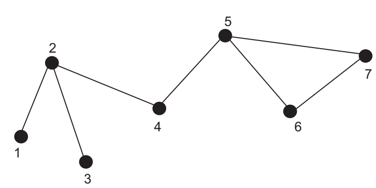
Notice that this graph is neither simple nor connected because of vertex 7. Also observe that in counting all degrees, every edge is counted twice (once for each vertex it is connected to), thus

$$\frac{1}{2}\sum_{v \in V} d_G(V) = |E| = \text{the total number of edges}$$

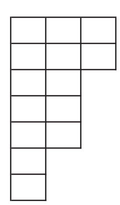
Definition 1.7. *If a connected graph G has k vertices and degrees $d_1 \geq d_2 \geq \ldots \geq d_k > 0$, then the degree sequence of G is $d(G) = (d_1, d_2, \ldots, d_k)$.*

Notice we are not necessarily assuming that $d_i = d(v_i)$ and clearly, the degree sequence is unique for every graph.

Example 1.11. *The following graph has degree sequence $d(G) = (3, 3, 2, 2, 2, 1, 1)$*

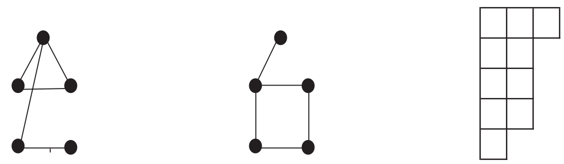


Observe that each degree sequence can be viewed as a partition and has an associated Young diagram, $Y(G)$, of shape $\lambda = d(G)$. For example, the degree sequence from the last example $d(G) = (3, 3, 2, 2, 2, 1, 1)$, can be represented as $Y(G)$:



There may be more than one graph associated with a Young diagram. Hence, the relationship between a graph and its Young diagram is not one to one.

Example 1.12. *Graphs G_1 and G_2 have the same degree sequence, so their associated Young diagrams are the same. However, G_1 is not isomorphic to G_2 since G_1 has one three cycle and G_2 has one four cycle. By isomorphic, we mean that there is a bijection between the vertices such that the induced mapping on the edges is also a bijection.*



$d(G_1) = (3,2,2,2,1)$ $d(G_2) = (3,2,2,2,1)$ $Y(G_1) = Y(G_2)$

Also, note that given a Young diagram, there may not be a graph associated with it.

Example 1.13. *For $\lambda = [4, 2, 2, 2, 1, 1]$ the Young diagram, $Y(\lambda)$ has no graph with the same degree sequence. (See Theorem 1.1)*

We are interested in those Young diagrams which can be represented with a graph. These are said to be **graphic**. Note that a partition of n which is a degree sequence has $\frac{1}{2}n$ edges in its representative graph, so $\sum_{v \in V} d(v)$ must be even. Thus a partition of an odd number is not graphic.

Theorem 1.1 (Ruch-Gutman Theorem)**.** *[2] Suppose λ is a partition of an even number n. Then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is graphic if and only if*

$$\sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{r} (\lambda_i^* - 1) \ \ \text{where } 1 \leq r \leq tr(\lambda).$$

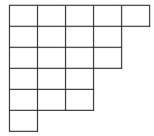
Proof. Suppose that λ is graphic and let λ^* be the conjugate partition. Number the verticies in the associated graph in such a way that λ_j= the number of edges at vertex j. Let Y be the associated Young diagram. In each row j of Y, place in increasing order the numbers of the verticies with edges to vertex j. Then the first column of Y contains all the 1's as well as a number greater than one in the $(1, 1)$ position. Hence $\lambda_1^* \geq \lambda_1 + 1$.

We continue this process until we get to the $k + 1$ column where $k = tr(\lambda)$ If $k \geq 2$, then the first two columns contain all the 1's and all the 2's and the first two positions in the second column contain numbers larger than 2. Hence $\lambda_1^* + \lambda_2^* \geq \lambda_1 + \lambda_2 + 2$. Continue this up to $tr(\lambda)$ proves half of the Ruch-Gutman Theorem. We will not prove the second half. We will prove a special case which pertains to the graphs which we will consider.

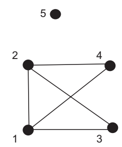
Definition 1.8. *Let $\lambda \vdash n$, where n is an even number. Then the partition λ is said to be a threshold partition if and only if $\lambda_i = \lambda_i^* - 1$ for $1 \leq i \leq tr(\lambda)$.*

Let λ be a threshold partition with associated Young diagram Y. Remove the first row and column from Y. The effect this has on λ is to delete λ_1, and to subtract 1 from all the remaining λ, some of which may now be 0. Call the partition σ. Ignoring the 0's, the σ and diagram are threshold. therefore they are are the degree sequence for a graph G. If there were any 0's in σ, then for each 0, add an isolated vertex. Now add one more vertex and construct an edge to each of the existing verticies. This graph has λ as it's degree sequence. Hence for threshold partitions at least, we have shown the existence of an associated graph.

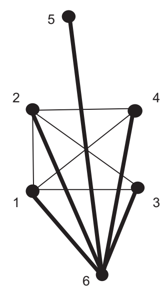
Example 1.14. *Let* $\lambda = (5, 4, 4, 3, 3, 1)$. *Then* Y *is*



Now σ *is (3,3,2,2,0). The associated graph is*



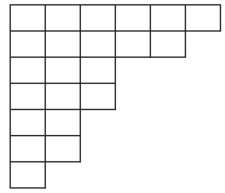
To this graph add a vertex and attach an edge to each other vertex.

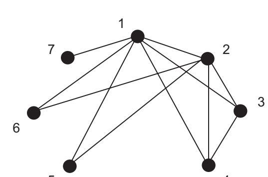


Definition 1.9. *A threshold graph is one whose degree sequence is a threshold partition.*

In other words, threshold graphs hold the property that up to the trace, each column is one more than its corresponding row.

Example 1.15. *Let* $\lambda = (6, 5, 3, 3, 2, 2, 1)$. *Then* λ *is a threshold partition whose Young diagram* $Y(\lambda)$ *and the associated (threshold) graph are shown below:*

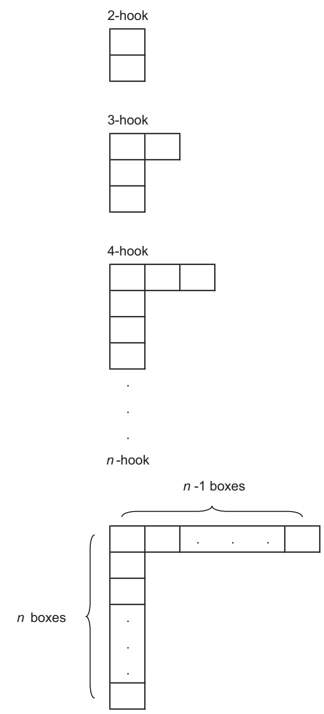




We will need one more definition to be used in the next chapter about diagrams.

Definition 1.10. *An n-hook is a column of n cells whose first row has n-1 cells.*

Example 1.16. *The following are the young diagrams for various n-hooks.*

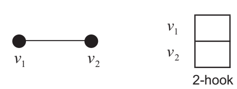


Chapter 2

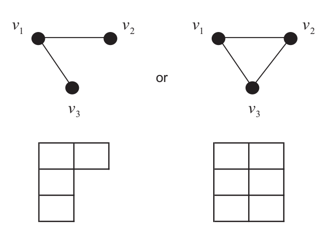
Counting the Number of Threshold Graphs

The number of threshold graphs can be counted by constructing their threshold diagrams. In this section, we will show by construction that the number of threshold graphs with n vertices is 2^{n-2}. At the end of this section, an alternate proof is given using combinatorial methods and Pascal's triangle.

We will start by constructing the smallest threshold graphs. First, define $v(n)$ =the number of threshold diagrams with n vertices. The smallest of these is $v(1) = 0$, which is called the empty diagram. It is the graph with one vertex and no edges. The next simplest graph is one with two vertices, $v(2)$, and there is only one graph. Also, the corresponding tableau is just a 2-hook as shown in the figure below.

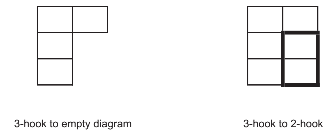


Notice that the threshold equality is met since $\lambda_1 = \lambda_1^* - 1$ or $(1 = 2 - 1)$, and the degree sequence is $d(G) = (1, 1)$. Also, notice that when $n = 2$, the number of threshold graphs is $2^{2-2} = 1$. Examining the case for $v(3)$, we can have the following possibilities shown in the figure below.



These both satisfy the threshold equality and also meets the condition that there are $2^{3-2} = 2$ threshold diagrams for three vertices.

The way to construct the next threshold diagram for $v(n)$ is to add an n-hook to all the previous diagrams, corresponding to vertices less than n. For example, to find $v(3)$, we will add a 3-hook to the diagram for $v(1)$ (the empty diagram), and add a 3-hook to



the diagram for $v(2)$ (the 2-hook). Doing this we get the following two diagrams, which are the two we already constructed.

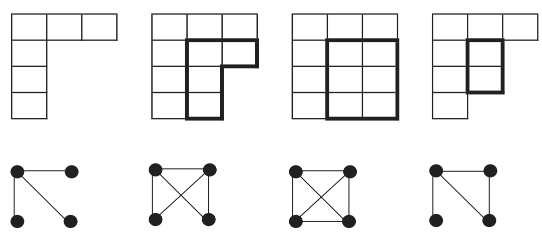
This is a valid construction since the threshold equality is satisfied for any n-hook, since it is a column of n-cells with a first row of n-1 cells and the trace of any n-hook is 1. Since the hook is added to a previous threshold diagram, the equality is still preserved up to the trace of the entire diagram. A short outline of a proof follows.

We define a previous partition as λ^{old}. Note that for a n-hook, $\lambda_1^* = n$ and $\lambda_1 = n-1$, therefore $\lambda_1^* = \lambda_1 + 1$. Furthermore, tr($\lambda$) =tr($\lambda^{old}$) + 1 and

$$\lambda_i^* = \lambda_{i-1}^{old*} + 1$$
$$\lambda_i = \lambda_{i-1}^{old} + 1 \text{ for } 2 \leq i \leq tr(\lambda)$$

which implies $\lambda_i^* = \lambda_i + 1$ for $1 \leq i \leq tr(\lambda)$.

Example 2.1. *The number of threshold diagrams for a graph with 4 vertices, $v(4)$, can be seen by adding a 4-hook to all previous diagrams. We can see below that the total number of threshold graphs and diagrams are $2^{4-2} = 2^2 = 4$.*



This method of construction proves the number of threshold diagrams for n vertices is 2^{n-2}. Because all diagrams are constructed from previous diagrams so we see that

$$\begin{aligned}
v(1) &= 0 \\
v(2) &= v(1) = 1 \\
v(3) &= v(1) + v(2) = v(1) + v(1) = 2 * v(1) = 2 * 1 = 2^{3-2} \\
v(4) &= v(1) + v(2) + v(3) = v(3) + v(3) = 2 * 2 = 2^{4-2} \\
&\vdots \\
v(n) &= 2 * v(n-1) = 2^{n-2}
\end{aligned}$$

2.0.1 Alternate Proof

We can also count the number of threshold graphs by referring to Pascal's triangle (Table 2.1).

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1
⋮				

Table 2.1: Pascal's Triangle

Recall that threshold diagrams for n vertices are formed by adding an n-hook to every threshold diagram with vertices less than n and the resulting trace is $\text{tr}(\lambda) = \text{tr}(\lambda^{old}) + 1$. Since the addition of this n-hook increases the trace by one, the number of diagrams of n vertices that have trace k is the number of diagrams that had trace $k-1$ for all diagrams with vertices less than n. This pattern can be summed up using binomial coefficient and is illustrated in Figure 2.2. More specifically, the number of diagrams of n vertices that have trace k is $\binom{n-2}{k-1}$ for $1 \leq k \leq n-1$. Then the total number of diagrams of n vertices is $\binom{n-2}{0} + \binom{n-2}{1} \cdots + \binom{n-2}{n-2} = 2^{n-2}$.

Vertices	Trace							
	1	2	3	4	5	⋯	n-1	sum
2	1	0	0	0	0	⋯	0	2^0
3	1	1	0	0	0	⋯	0	2^1
4	1	2	1	0	0	⋯	0	2^2
5	1	3	3	1	0	⋯	0	2^3
6	1	4	6	4	1	⋯	0	2^4
⋮								
n	$\binom{n-2}{0}$	$\binom{n-2}{1}$	$\binom{n-2}{2}$	$\binom{n-2}{3}$	$\binom{n-2}{4}$	⋯	$\binom{n-2}{n-2}$	2^{n-2}

Table 2.2: Pascal's Triangle related to Threshold Diagrams

Now we need to show that 2^{n-2} is also the number of threshold graphs for n vertices by showing there is a one-to-one correspondence between threshold diagrams and their graphs. We know from the previous chapter that every graph has only one degree sequence and therefore only one Young diagram. So given a threshold graph there is only one threshold diagram corresponding to it. However, as seen in the last chapter, in general it is not true that there is only one graph for every Young diagram. But, using inductive reasoning, we can show that given a *threshold* diagram, then there is only one corresponding threshold graph.

For the first case, $n = 2$ we showed earlier that there is only one threshold diagram for two vertices. Since the graph involves only one edge between the two vertices, this graph is unique. Now lets assume this property is true for n vertices. So for n vertices, each threshold diagram has a unique threshold graph. Recall that to construct the diagrams for $n+1$ vertices, an $n+1$ hook is added to each diagram with vertices less than $n+1$. Graphically, this is the same as the addition of a vertex that is connect to every other vertex of the graph, and there is only one way to do this as seen in Figure 2.1. Since the relationship between the graph and the diagram for n vertices is unique and the $n+1$-hook addition is unique, the threshold diagram with $n+1$ vertices has only one corresponding threshold graph. Hence the relationship between them is one-to-one.

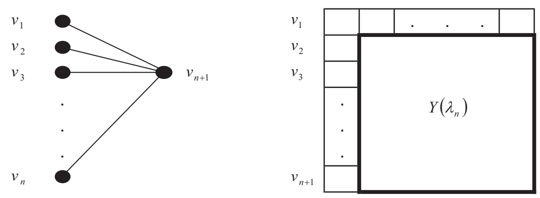


Figure 2.1: The threshold diagram with n+1 vertices and its only corresponding threshold graph

Chapter 3

Longest and Shortest Subsequences

Let $\lambda = [\lambda_1, \ldots, \lambda_n]$ be a partition of n. Recall that a Young diagram T is called a standard tableau if the integers $1, 2, \ldots, |\lambda|$ are placed injectively in the cells of T such that the elements in each row and column are increasing. For the remainder of the paper, we will drop the use of cells when drawing the tableaux.

Definition 3.1. *A Young diagram of shape λ is called a semistandard tableau if the cells are filled with the integers, repetitions allowed, such that the elements in the columns are strictly increasing, and the elements in the rows are non-decreasing.*

Definition 3.2. *The **content** $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ of a semi-standard tableau T has integer $\mu_j =$ the number of j's in T.*

Example 3.1. *Let* $T = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & & \end{matrix}$ *and* $S = \begin{matrix} 1 & 3 & 3 \\ 5 & 5 & 7 \\ 7 & & \end{matrix}$. *Then T is standard of shape $\lambda = [3, 3, 1]$ and S is semi-standard of shape $\lambda = [3, 3, 1]$ with content $\mu = (1, 0, 2, 0, 2, 0, 2)$.*

3.1 Method of Insertion

We now describe the operation of inserting an integer, x, into a semi-standard tableau T. Insertion begins in the first row and then:

1. If x is at least as large as each element in the first row, place x at the end of the row and stop.

2. Otherwise, working east to west, find the last element y in the row that is larger than x. Replace y by x and now repeat this process with y in the second row. Continue until some element falls into case one.

Example 3.2. *Let* $T = \begin{matrix} 1 & 2 & 2 & 3 & 5 & 5 & 6 \\ 2 & 3 & 4 & 4 & 6 & & \\ 4 & 4 & 5 & & & & \\ 6 & & & & & & \end{matrix}$ *and let $x = 2$. Now inserting x into T is demonstrated with the element on the right placed in the position marked by an asterisk* *as*

$$\begin{array}{ccccccccc} 1 & 2 & 2 & 2^* & 5 & 5 & 6 & \leftarrow & 2 \\ 2 & 3 & 3^* & 4 & 6 & & & \leftarrow & 3 \\ 4 & 4 & 4^* & & & & & \leftarrow & 3 \\ 5^* & & & & & & & \leftarrow & 5 \\ 6^* & & & & & & & \leftarrow & 6 \end{array}.$$

Notice that

1. The replaced elements increase.
2. The path of the replaced elements is Southwest
3. The next tableau is semi-standard.

These results always hold for insertion. Suppose that x is inserted into the first row of T and replaces y. The pertinent elements look like

$$\begin{array}{ccccc} s & y & t & \leftarrow & x \\ & z & & & \end{array}$$

where $s \leq y \leq t$, $x < y < z$, and $s \leq x$. After inserting x in the first row, y is inserted into the second row as follows:

$$\begin{array}{ccccc} s & x & t & & \\ & z & & \leftarrow & y \end{array}$$

and either

$$\begin{array}{ccc} s & x & t \\ & y & \end{array} \text{ or } \begin{array}{ccc} s & x & t \\ y & z & \end{array}$$

where $y \ldots z$ means that y is West of z. It is seen here that properties 1, 2, and 3 all hold.

Definition 3.3. *The insertion path is the collection of cells that are involved in the insertion process.*

In Example 3.2, the insertion path is indicated by the asterisks.

A special case of insertion is when the elements in T and the element x are all distinct. The same rules apply, but $\leq$ is replaced by $<$.

Example 3.3. *Let* $T = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & & \\ 6 & & \end{array}$ *and* $x = 4$. *Then* $\begin{array}{ccccc} 1 & 3 & 4^* & \leftarrow & 4 \\ 2 & 5^* & & \leftarrow & 5 \\ 6 & & & & \end{array}$.

3.2 Permutations

For each permutation $\pi = \begin{pmatrix} 1 & \ldots & n \\ i_1 & \ldots & i_n \end{pmatrix}$ there is an associated standard tableau. First, write π by just using the second line $i_1 \ldots i_n$. This is called one-line notation. Now insert $i_1, \ldots, i_n$ in the order $i_1, i_2, \ldots, i_n$.

Example 3.4. *If* $\pi = (3127465)$ *then the sequence of tableaux are*

$$
\begin{array}{l}3\end{array}\qquad
\begin{array}{l}1\\3\end{array}\qquad
\begin{array}{ll}1&2\\3&\end{array}\qquad
\begin{array}{lll}1&2&7\\3&&\end{array}\qquad
\begin{array}{lll}1&2&4\\3&7&\end{array}\qquad
\begin{array}{llll}1&2&4&6\\3&7&&\end{array}\qquad
\begin{array}{llll}1&2&4&5\\3&6&&\\7&&&\end{array}
$$

Denote the final tableau by $P(\pi)$ and notice that the permutation $\pi_1 = (7361245)$ yields the same tableau, so $P(\pi) = P(\pi_1)$ which we denote by P. Now π_1 has a special relation with P. Start with the last row of P and working up by rows, list the elements. In the previous example, we list 7, then 7 3 6, and finally 7 3 6 1 2 4 5, which is π_1. Given any standard tableau T, this process yields a permutation π such that $P(\pi) = T$.

This remark can be seen by observing that the insertion process for this π first yield the last row of T, which is bumped with the second row by what will become the next to last row of T. This process continues to obtain T, and the permutation constructed from T is called the **row word** of T.

Caution: Not every permutation is a word for some T*. For example, why does* $\pi = (132)$ *not work?*

Since different permutations can have the same tableau, the mapping from $\pi \to P(\pi)$ is not a bijection from $S_n = \{$the set of all permutations on the numbers 1 to n$\}$ into the collection of standard tableaux with $n-$cells. The mapping is surjective since the word π for T yields $P(\pi) = T$. We will show that there is a bijection from S_n onto $\Omega(n) = \{$the set of all pairs of standard tableaux of the same shape with n-cells$\}$.

Given π, construct $P(\pi)$ by the insertion process. In each step, a new cell is formed as in Example 3.4. A second tableau $Q(\pi)$ is also constructed simultaneously by successively adding a new cell in the same position as in the construction of $P(\pi)$. When the $i'th$ cell is constructed, an i is placed in it.

Example 3.5. *Let* $\pi = (3127465)$ *as in the previous example then the sequence of tableaux* P *and* Q *are*

$$
P(\pi):\ \begin{array}{l}3\end{array}\qquad
\begin{array}{l}1\\3\end{array}\qquad
\begin{array}{ll}1&2\\3&\end{array}\qquad
\begin{array}{lll}1&2&7\\3&&\end{array}\qquad
\begin{array}{lll}1&2&4\\3&7&\end{array}\qquad
\begin{array}{llll}1&2&4&6\\3&7&&\end{array}\qquad
\begin{array}{llll}1&2&4&5\\3&6&&\\7&&&\end{array}
$$

$$
Q(\pi):\ \begin{array}{l}1\end{array}\qquad
\begin{array}{l}1\\2\end{array}\qquad
\begin{array}{ll}1&3\\2&\end{array}\qquad
\begin{array}{lll}1&3&4\\2&&\end{array}\qquad
\begin{array}{lll}1&3&4\\2&5&\end{array}\qquad
\begin{array}{llll}1&3&4&6\\2&5&&\end{array}\qquad
\begin{array}{llll}1&3&4&6\\2&5&&\\7&&&\end{array}
$$

Now by assigning π *to the pair* $(P(\pi), Q(\pi))$*, we have created a bijection as we will explain.*

There is also an inverse process to insertion. Suppose that x is inserted into a semi-standard tableau T and T' is obtained. The steps required to recover T from T' are as follows. Suppose that the final cell in the insertion path of x contains μ. Note that μ

is at the end of a row in T'. To reverse the insertion of x we retrace the insertion path. First, μ replaces the last element, ν, that is less than μ. Then in the next row North, repeat the process with ν. Continue this process until the first row is reached and the final element replaces will be x. Hence, given $P(\pi)$ and the order in which the insertion paths end, given by $Q(\pi)$, we can invert the construction of $P(\pi)$ to get π.

Example 3.6. *Here is the reverse of the steps in example 3.4. We begin with*

$$P(\pi) = \begin{matrix} 1 & 2 & 4 & 5 \\ 3 & 6 & & \\ 7 & & & \end{matrix} \qquad \textit{and}\, Q(\pi) = \begin{matrix} 1 & 3 & 4 & 6 \\ 2 & 5 & & \\ 7 & & & \end{matrix}$$

The largest element in $Q(\pi)$ indicates the cell to start with in $P(\pi)$. So the 7 in the $(3,1)$ position in $P(\pi)$ is moved into the second row, replacing 6, which moves into the first row replacing 5. After one step $P(\pi)$ becomes $\begin{matrix} 1 & 2 & 4 & 6 \\ 3 & 7 & & \end{matrix}$ *and $Q(\pi)$ becomes* $\begin{matrix} 1 & 3 & 4 & 6 \\ 2 & 5 & & \end{matrix}$. *The full inverse process is as follows:*

Step i	P_i	Q_i	π
7	1 2 4 5 3 6 7	1 3 4 6 2 5 7	-
6	1 2 4 6 3 7	1 3 4 6 2 5	(5)
5	1 2 4 3 7	1 3 4 2 5	(65)
4	1 2 7 3	1 3 4 2	(465)
3	1 2 3	1 3 2	(7465)
2	1 3	1 2	(27465)
1	3	1	(127465)
0	-	-	(31274565)

In summary, the process that we have described gives a bijection $\pi \leftarrow (P(\pi), Q(\pi))$ from S_n onto $\Omega(n)$.

As we have seen, two different permutations can have the same insertion tableau. Therefore, given such a tableau that is constructed from a permutation, it is impossible to know which permutation from which it came. To remedy this process, Robinson constructs a placement tableau for the permutation. When the i_{th} element in the permutation is inserted in the construction of P, a new cell is formed. We keep track of the positions of these new cells in a tableau Q. Formally, given permutation π, a pair of tableaux, (P, Q) is constructed, P by insertion and Q by placement. For example, let $\pi = (3127465)$. The sequence of steps in constructing $(P(\pi), Q(\pi))$ is as follows

$P(\pi):$	3	1 3	1 2 3	1 2 7 3	1 2 4 3 7	1 2 4 6 3 7	1 2 4 5 3 6 7

$Q(\pi):$	1	1 2	1 3 2	1 3 4 2	1 3 4 2 5	1 3 4 6 2 5	1 3 4 6 2 5 7

Hence π is assigned to $(P(\pi), Q(\pi))$, a process which we claim is a bijection between the set of permutations of length n and pairs of standard tableaux of the same shape with n cells. The inverse process is as follows. Find the cell with the largest integer in Q and remove this cell from Q. The integer, x, in that position in P is removed by retracing the insertion path. Note that x is in the last element in some row. As the process begins, the cell in P which contains x is removed. Move up to the next row and find, moving East, the largest integer, y, that is less than x. Then y is removed from that row and move up to the next row to repeat the process. The final integer bumped from the first row goes at the East end of the permutation which is in one line notation. The process continues with the remaining largest integer in Q. This process is the inverse of the construction of $(P(\pi), Q(\pi))$ and the desired bijection is established.

If the restriction that elements in the sequence are distinct is removed, then the same process shows that the is a bijection between sequences, ω of length n of positive integers from 1 to n, with repetitions allowed, and pairs, $(P(\omega), Q(\omega))$, of tableaux of the same shape with n cells and entries from 1 to n, P semi-standard and Q standard. In this form the bijection is the Robinson-Schensted correspondence.

Theorem 3.1. *Let f^λ be the number of standard tableaux of shape λ, $|\lambda| = n$. The bijection gives*

$$n! = \sum_\lambda (f^\lambda)^2$$

where the sum is over all partitions of n.

Example 3.7. *Let $n = 3$. Then*

Partition λ	f^λ	List of Standard Tableaux
[3]	1	1 2 3
[2,1]	2	1 2 3 and 1 3 2
[1,1,1]	1	1 2 3

Then $3! = 1^2 + 2^2 + 1^2$.

The preceding process can be generalized. Suppose that π is allowed to have repetitions. Then π is no longer a permutation in one line notation, π is now called a word. We have described how to construct a semi-standard tableau, $P(\pi)$, from any word using the insertion process. $Q(\pi)$ is constructed as before, keeping track of the order in which new cells are added. The inverse procedure is also valid. Hence consider the set S of all words of length n with elements j, where $0 \leq j \leq t$ for a fixed t. The bijection $s_n \leftarrow \Omega(n)$ becomes a bijection from S to (P, Q), where P and Q are tableaux of the same shape with n cells and P is semi-standard while Q is standard.

Definition 3.4. *The number of semi-standard tableaux of shape λ and content μ, denoted by $K_{\lambda,\mu}$, is called the Kostka number for λ and μ.*

Example 3.8. *Let $\lambda = [3, 2]$ and $\mu = [2, 1, 1, 1]$. Then the following are all the possible semi-standard tableaux of shape λ and content μ.*

$$\begin{matrix} 1 & 1 & 2 \\ 3 & 4 & \end{matrix} \qquad \begin{matrix} 1 & 1 & 3 \\ 2 & 4 & \end{matrix} \qquad \textit{and} \quad \begin{matrix} 1 & 1 & 4 \\ 2 & 3 & \end{matrix}$$

Hence $K_{\lambda,\mu} = 3$.

The following theorem is a generalization of Theorem 3.1 and the proof uses the bijection between words of length n using integers between 1 and t and tableaux (P, Q) of the same shape having n cells. Again, P is semi-standard and Q is standard and the cells in P contain integers between 1 and t.

Theorem 3.2.

$$|S| = \sum_{\lambda} \sum_{\mu} (f^{\lambda}) K_{\lambda,\mu}$$

where the first sum is over all the partitions of n and the second sum is over the content μ with elements from 1 to t.

Exercise 3.1. *Let λ and μ be partitions of n, where $\lambda = [\lambda_1, \ldots, \lambda_k]$ and $\mu = [\mu_1, \ldots, \mu_k]$ where we add 0's at the end of λ or μ in order so that they have the same length. Define $\lambda \geq \mu$ if $\lambda_1 + \cdots + \lambda_j \geq \mu_1 + \cdots + \mu_j$ for $j = 1, 2, \ldots, k$, and also define $\lambda > \mu$ if $\lambda \geq \mu$ but $\lambda \neq \mu$. Then*

1. *Prove that if $\mu > \lambda$, then $K_{\lambda,\mu} = 0$.*
2. *Prove that if $\mu = \lambda$, then $K_{\lambda,\mu} = 1$.*
3. *Find w if* $P(w) = \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \\ 3 & & & \end{matrix}$ *and* $Q(w) = \begin{matrix} 1 & 2 & 5 & 8 \\ 3 & 4 & 7 & \\ 6 & & & \end{matrix}$.
4. *Find $P(w)$ and $Q(w)$ for $w = (52165535)$ and for $w = (145362432)$.*
5. *Find $K_{\lambda,\mu}$ for $\lambda = [4, 3, 1]$ and $\mu = [3, 2, 1, 2]$.*

3.3 Subsequences of Words

Let w be a word. In this section, we will find the length of the longest non-decreasing subsequence of w and also the length of the longest decreasing subsequence of w. When w is a permutation (no repeated numbers), then the method presented finds the length of the longest increasing subsequence of w. We begin by considering permutations.

Consider the case when w is a word of a standard tableau T, of shape $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$. There is an increasing subsequence of w of length λ_1 using the elements from the first row of T. Let σ be an increasing subsequence of w. We claim that σ can contain no more than one element from each column of T.

Suppose that a and b are elements of T and are in the same column, C, of T and that a comes before b in σ. Then $a < b$. But the construction of w is such that rows of T appear in ascending order in w. Hence $b < a$, which is a contradiction. Consequently, the elements of σ come from different columns of T and the length of σ is at most λ_1. Thus the longest increasing subsequence of w has length λ_1.

Now consider the length of the longest decreasing subsequence of w. The elements in the first column of T are in increasing order, and they are in reverse order in w. Hence they form a decreasing subsequence of w of length k, the number of rows in T. Let σ be an decreasing subsequence of w. Each row of T has its elements in adjacent positions in w with the order preserved. Hence no two elements of σ come from the same row and σ can have length at most k. Therefore, the length of the longest decreasing subsequence of w is k.

For general sequences, let π be a sequence and $P(\pi)$ be the tableau formed by insertion and w be the word associated with $P(\pi)$. We claim that the longest increasing (or decreasing) subsequences of π and w have the same length.

Example 3.9. *Let $\pi = (68123547)$.*

Then $P(\pi) = \begin{array}{lllll} 1 & 2 & 3 & 4 & 7 \\ 5 & 8 \\ 6 \end{array}$ *and then* $w = (65812347)$.

A longest increasing subsequence for w is 12347 and that is also true for π. The longest decreasing subsequence of w is 654 and that is also true for π. However, it is not always the case that the same subsequences work for both π and w, but the lengths will be the same.

Looking more closely at insertion, in the previous example we inserted 4 into $T = \begin{array}{llll} 1 & 2 & 3 & 5 \\ 6 & 8 \end{array}$. We will write the process as $\begin{array}{llllll} 1 & 2 & 3 & 5 & . & 4 \\ 6 & 8 \end{array} = T \cdot 4$. Inserting one more move at a time shows that

$$T \cdot 4 \to \begin{array}{lllll} 1 & 2 & 3 & 5 & 4 \\ 6 & 8 \end{array} \to \begin{array}{lllll} 1 & 2 & 5 & 3 & 4 \\ 6 & 8 \end{array} \to \begin{array}{lllll} 1 & 5 & 2 & 3 & 4 \\ 6 & 8 \end{array} \to \begin{array}{lllll} 5 & 1 & 2 & 3 & 4 \\ 6 & 8 \end{array} \to$$

$$\begin{array}{llll} 1 & 2 & 3 & 4 \\ 6 & 8 & 5 \end{array} \to \begin{array}{llll} 1 & 2 & 3 & 4 \\ 6 & 5 & 8 \end{array} \to \begin{array}{llll} 1 & 2 & 3 & 4 \\ 5 & 8 \\ 6 \end{array} = T_1$$

Let us consider this process using the word $w = (681235)$. Then $w.4 = (681235.4) = (6812354) \to (6812534) \to (6815234) \to (6851234) \to (6581234) = w_1$ which is the word for T_1.

These moves have been accomplished by interchanging adjacent integers under the following conditions:

For $x < y < z$

1. $yxz = yzx$
2. $xzy = zxy$

These relations are called **Knuth** relations of the first and second kind. Inserting s into tableau T yields tableau T' where the words of T and T' are related by a sequence of Knuth relations. In the previous example, the first three moves were by a Knuth relation of the second kind and the last move is by a Knuth relation of the first kind. In particular, the following table shows the moves:

Move	x	y	z
1	3	4	5
2	2	3	5
3	1	2	5
4	5	6	8

We will show that when two words differ by a Knuth relation, then the lengths of the longest increasing (or decreasing) subsequences are invariant.

Consider the increasing case and a Knuth relation of the first kind. Let w be a word, a part of which is $ayxzb$ with $x < y < z$ and where a and b are not necessarily adjacent to y and z. Let w' be the word with yxz replaced by yzx (the Knuth relations of the first kind). Let σ be a maximal increasing subsequence of w which contain a and b but no other element of w except some of the y, x and z. If none of the x, y, z are in σ, then the Knuth relation has no effect. If $ayzb$ are (now adjacent) elements in σ then the same sequence is a maximal increasing subsequence of w'.

If $axzb$ is part of σ, then $ayzb$ is part of a maximal increasing subsequence of w', which is the same as σ except for the substitution of y for x. Hence Knuth relations of the first kind do not alter the lengths of the longest increasing subsequence of w and w'. The same is true for Knuth relations of the second kind and also for lengths of decreasing subsequences.

In summary, given a permutation π, we can form $P(\pi)$ and let w be the associated word for $P(\pi)$. Then w and π are Knuth equivalent, where their longest increasing (or decreasing) subsequence are the same. Thus the length of the first row (or column) of $P(\pi)$ gives the length of the longest increasing (or decreasing) subsequences of w, hence of π. The result is stated in the following theorem.

Theorem 3.3. *Let π be a permutation. The length of the longest increasing subsequence of π is the length of the first row of $P(\pi)$ and the length of the longest decreasing subsequence of π is the length of the first column of $P(\pi)$.*

Example 3.10. *Let* $\pi = (925746138)$. *Then* $P(\pi) = \begin{matrix} 1 & 3 & 6 & 8 \\ 2 & 4 & & \\ 5 & 7 & & \\ 9 & & & \end{matrix}$. *The longest increasing subsequence and longest decreasing subsequence both have length 4. Note that the first row of* $P(\pi)$ *is not a subsequence of* π, *nor is the first column. These results only give the length of the subsequence, not the subsequence itself.*

If we consider words instead of of just permutations, we recover similar results with similar arguments. Starting with word w, construct $P(w)$, which is a semi-standard tableau. When the word τ of $P(w)$ is formed, the length of the first row of $P(w)$ is the length of the longest non-decreasing subsequence of τ and the length of the first column of $P(w)$ is the length of the longest decreasing subsequence of τ. As before, insertion corresponds to Knuth relations, slightly altered. Now they are:

1. $yxz = yzx$ when $x < y \leq z$
2. $xzy = zxy$ when $x \leq y < z$

As before, Knuth equivalent words leaves the lengths we are considering invariant. Thus w and τ are Knuth equivalent and the shape of $P(w)$ gives the desired information.

Theorem 3.4. *Let* w *be a word and* $P(w)$ *be the insertion tableau. The length of the longest non-decreasing subsequence of* w *is the length of the first row of* $P(w)$ *and the length of the longest decreasing subsequence of* w *is the length of the first column of* $P(w)$.

Exercises Given $w = (523527154263)$, find the length of

1. The longest non-decreasing subsequence of w.
2. The longest decreasing subsequence of w.
3. The longest non-increasing subsequence of w.
4. The longest increasing subsequence of w.

We consider a generalization of these results.

Given a word w, then k subsequences are disjoint if no element from w is in more than one of the subsequences. Modifying the previous arguments, it can be shown that $k-$non-decreasing disjoint subsequences can be found such that the total number of elements in the subsequences is equal to the number of elements in the first k rows of $P(w)$ and that this is the maximum possible. Similar remarks hold for all of our results.

Exercise 3.2. *In the previous exercise, what is the length of the longest pair of non-decreasing (or decreasing) disjoint subsequence of* w*? What is the length of longest pair of non-increasing (or increasing) disjoint subsequences of* w*?*

Chapter 4

Counting Certain Sets of Matrices

Let $A = (a_{i,j})$ be an $m \times n$ matrix with non-negative integer entries. Then $r_i = \sum_{j=1}^{n} a_{i,j}$ is called the i'th row sum of A and $c_j = \sum_{i=1}^{m} a_{i,j}$ is called the j'th column sum of A. We will develop a method which finds the number of A which have any given sequence $r_1, ...r_m, c_1, ..., c_n$ of row and column sums.

For each A, construct a two row array ω with columns $\begin{pmatrix} i \\ j \end{pmatrix}$ occurring $a_{i,j}$ times and $\begin{pmatrix} i \\ j \end{pmatrix}$ appearing before $\begin{pmatrix} k \\ l \end{pmatrix}$ if $i < k$ or $i = k$ and $j < l$. Such an array is said to be in lexicographic order. There are $z = \sum r_i = \sum c_j$ columns in ω.

Example 4.1. *Let*

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 4 & 2 \end{pmatrix}.$$

Then the associated array is $\omega = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$.

For a fixed m and n and row and column sums, there is a bijection between matrices and associated arrays. These arrays are generalizations of the words that we considered previously where the first row consists of integers 1 through n, each appearing once. Using the the same methods as before, a pair of tableaux is constructed. Now each tableau is semistandard. In particular, given ω, $P(\omega)$ is constructed by insertion and $Q(\omega)$ is constructed by placement. Thus the i'th element in the second row ω is inserted into a cell of $P(\omega)$ and the i'th element in the first row of ω is placed in the corresponding cell in $Q(\omega)$.

Example 4.2. *Let*

$$\omega = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 & 2 & 5 & 2 & 1 \end{pmatrix}.$$

The construction of $P(\omega)$ and $Q(\omega)$ proceeds as follows.

P	Q
1	1
12	11
122	111
112 2	111 2
1122 2	1112 2
11225 2	11122 2
11222 25	11122 23
11122 22 5	11122 23 4

Both final tableaux are semistandard

We will now describe an inverse procedure. Start with semistandard tableaux, P and Q. We find an array ω such that $P(\omega) = P$ and $Q(\omega) = Q$. We now explain the order in which elements from P and Q are removed. If there is a single largest element in Q, then it's cell determines the cell where the inverse procedure begins. The element in that cell in P is removed using the same inverse procedure as in the case of words. If there are several largest elements, u, in Q then choose the cell furthest east that contains u. Performing the inverse insertion process on the element in the corresponding cell in P removes an element v from P. Then $\omega = \begin{pmatrix} u \\ v \end{pmatrix}$ is the final column in ω. The process is repeated until ω is constructed.

Example 4.3. *The step by step inverse procedure in the last example gives*

$$\omega = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
$$\omega = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$$
$$\omega = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \end{pmatrix}$$
$$\omega = \begin{pmatrix} 2 & 2 & 3 & 4 \\ 2 & 5 & 2 & 1 \end{pmatrix}$$
$$\omega = \begin{pmatrix} 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 5 & 2 & 1 \end{pmatrix}$$
$$\omega = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 4 \\ 2 & 1 & 2 & 5 & 2 & 1 \end{pmatrix}$$
$$\omega = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ 2 & 2 & 1 & 2 & 5 & 2 & 1 \end{pmatrix}$$
$$\omega = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 & 2 & 5 & 2 & 1 \end{pmatrix}$$

Lemma 4.1. *Let a and b be in a semistandard tableau, P, with a south West of b. If the inverse insertion procedure is preformed on b, removing x, and then on a, removing y, then $x \geq y$.*

Proof. The removal path of a is row by row West of the removal path of b and the result follows. □

As a consequence of lemma 4.1, the array formed by the inverse procedure is lexicographically ordered. Therefore the inverse procedure is indeed the inverse to insertion. Thus there is a bijection between arrays and pairs of semistandard tableaux.

Each $n \times m$ matrix with non negative integer entries and prescribed row sums $r = [r_1, ...r_m]$ and column sums $c = [c_1, ..., c_n]$ gives a lexicographically ordered array ω with $z = \sum r_i = \sum c_j$ columns whose first row contains r_i i's and the second row contains c_j j's. Then ω determines a pair of semistandard tableaux, $(P(\omega), Q(\omega))$ of the same shape which come from a partition of z. The first tableau has content has content c and the second tableau has content r. Both correspondences are bijections.

Theorem 4.1. *Fix m, n, $r = [r_1, ...r_m]$ and $c = [c_1, ..., c_n]$ The number of $m \times n$ matrices with non negative integer entries and row sums given by r and column sums given by c is $\sum K_{\lambda,c}K_{\lambda,r}$ where the K's are Kostka numbers, the number of semistandard tableaux of shape λ and content c or r and λ runs over all partitions of z.*

Example 4.4. *Consider all* 2×3 *matrices with* $r = [r_1, r_2] = [2, 4]$ *and* $c = [c_1, c_2, c_3] = [2, 1, 3]$. *Then* $z = 6$. *We find all semistandard tableaux of shape a partition of* z *and content* r *and* c *respectively.*

λ	r= [2,4]	c =[2,1,3]	
[6]	1 1 2 2 2	1 1 2 3 3 3	
[5,1]	1 1 2 2 2 2	1 1 2 3 3 3	1 1 3 3 3 2
[4,2]	1 1 2 2 2 2	1 1 2 3 3 3	1 1 3 3 2 3
[4,1,1]	none and none for the remainder of the partitions.		

Hence for $\lambda = [6]$ *the product of Kostka numbers is 1, for* $\lambda = [5, 1]$ *the product of Kostka numbers is 2 and for* $\lambda = [4, 2]$ *the product of Kostka numbers is 2. The total number of matrices with the prescribed row and column sums is 5.*

We can also find the matrices. They are

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \text{ and}$$

$$A_5 = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

Exercise 4.1. *Find the number of* 3×3 *matrices with nonnegative integer entries with row sums* $[3, 3, 2]$ *and column sums* $[5, 1, 2]$.

4.1 Counting Standard and Semistandard Tableaux

There are remarkable formulas for counting the number of standard and semistandard tableaux. Let Y be the Young diagram associated with partition λ. Define a hook at any cell, c, in the (i, j) position in Y to be c along with all cells east in the row containing c and all cells south in the column containing c. Let $h(i, j)$ be the number of cells in the hook. $h(i, j)$ is called the hook length. Notice that these hooks are different than the ones in Chapter 2.

Theorem 4.2. *The number of standard tableaux for the partition λ, where $|\lambda| = n$ is*

$$\frac{n!}{\textit{product of all hook lengths in the diagram}}.$$

Example 4.5. *The number of standard tableaux for $\lambda = [5, 4, 3]$ is*

$$\frac{12!}{7 \times 6 \times 5 \times 3 \times 1 \times 5 \times 4 \times 3 \times 1 \times 3 \times 2 \times 1}.$$

At each cell we list the associated hook length.

7 6 5 3 1
5 4 3 1
3 2 1

A cute justification of the formula, assuming that the probabilities are independent (which they aren't), is the following. There are $n!$ ways to distribute the integers into cells. A distribution will be standard if each hook has it's smallest integer at the original cell; that is, where the hook is. The probability that a hook has this property is $\frac{1}{\text{it's hook length}}$. Hence

$$\frac{\text{the number of standard tableaux}}{n!} = \frac{1}{\text{product of the hook lengths}}.$$

There is also a result which count the number of semistandard tableaux for partition λ where the entries come from the first m positive integers.

Theorem 4.3. *The number of semistandard tableaux of shape λ whose entries are from the first m positive integers is*

$$\frac{\prod_{(i,j)\in\lambda}(m + j - i)}{\prod_{(i,j)\in\lambda} h(i, j)}.$$

Example 4.6. *Let $\lambda = [4, 3, 2]$ and $m = 5$.*

There is a simple device used to obtain the numerator in the formula. Fill the diagram for λ with m down the diagonal and then, starting on the diagonal, going East along each row, increase the entries by 1 each step and going South along each column, decrease the entries by one each step. These are the entries for the numerator

5 6 7 8
4 5 6
3 4

As before, we use the Young diagram to record the hook lengths.

6 5 3 1
4 3 1
2 1

The number of semistandard tableaux for λ and m is the product of all elements in the first table divided by the product of all elements in the second table. That quotient is 1120.

Chapter 5

Symmetry in tableaux $P(\lambda)$ and $Q(\lambda)$

There is a remarkable symmetry in the tableaux $P(\lambda)$ and $Q(\lambda)$. For λ a permutation this was shown by Viennot and for λ an array it was generalized by Knuth. In either case the result is that interchanging the rows in λ has the effect of interchanging $P(\lambda)$ and $Q(\lambda)$. In particular, if λ is a permutation which is an involution; that is, λ is equal to its inverse, then $P(\lambda) = Q(\lambda)$. In the case of arrays, an interpretation is that if the associated matrix is symmetric, then $P(\lambda) = Q(\lambda)$.

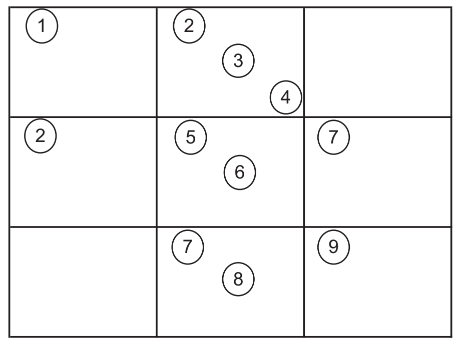
For permutations, a consequence of the symmetry is that there is a bijection between permutation involutions of length n and standard tableaux of size n. There is a recursion formula for the number, $a(n)$, of permutation involutions of size n . Let $S = (1, \ldots, n)$. As a product of disjoint cycles, a permutation involution has no cycles of length greater than two. To construct the recursion, consider two cases. Either n is in a two cycle or n is in a one cycle. The set of those in the second case is bijectively the same as the involutions on $(1, \ldots, n-1)$. Hence there are $a(n-1)$ of this type. In the first case, n is in a two cycle with some j. For each j, these involutions are constructed by taking each permutation involution on $(1, \ldots, j-1, j+1, \ldots, n-1)$ and affixing (j, n). The total number so obtained for each j is $a(n-2)$ and for all j is $(n-1)a(n-2)$. Therefore

$$a(n) = a(n-1) + (n-1)a(n-2).$$

Thus, the number of standard tableaux of size n is given by this formula. The first few values are 1, 2, 4, 10, 26.

We will discuss the symmetry theorem for the general case of arrays. For an array, ω, P_ω and Q_ω can be constructed by a different method called matrix ball construction [1]. Begin by constructing the matrix $A = (a_{i,j})$ as before, $a_{i,j}$ is the number of pairs $\begin{pmatrix} i \\ j \end{pmatrix}$ in ω. Now we construct a matrix of balls where $a_{i,j}$ balls are placed in the (i, j) position. In each matrix position slant the balls in a SouthEast direction. Number the balls by starting with a 1 in the most NorthWest position. In a matrix position with more than one ball, number the balls with consecutively increasing integers as moving SouthEast. On leaving a matrix position, move horizontally and vertically, and number balls with the smallest integer which has not been used in any northwest position.

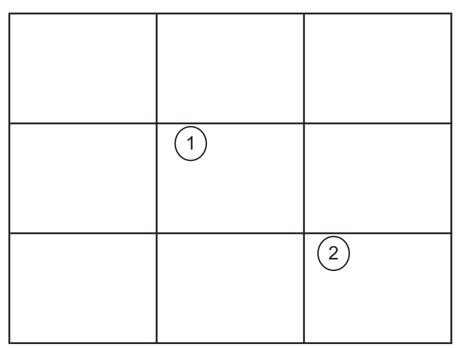
Example 5.1. *Let* $\omega = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 2 & 1 & 2 & 2 & 3 & 2 & 2 & 3 \end{pmatrix}$*. Then* $A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$ *and the matrix ball* B *is*



Clearly A *can be recovered from* B.

We now construct tableaux $P = P(B)$ and $Q = Q(B)$. In the first row in P, for each integer in B place the smallest number of the column number in B where there is a ball containing that integer. Likewise, in the first row in Q place the smallest row number which contains the integer.

In example 5.1, the first row of P is $1, 1, 2, 2, 2, 2, 2, 2, 3$ and of Q is $1, 1, 1, 1, 2, 2, 2, 3, 3$. To continue, a new ball matrix is constructed. For each integer k appearing in B, do the following. At each appearance of k, draw a ray South in that column and a ray East in that row. For each row ray, find the intersection with a column ray and place a ball in that position. Evidently, for each k, the number of new balls is one less than the number of balls labeled k in B. Now label the balls using the same process as before. In the example 5.1, the new matrix is



The second rows in P and Q are found as were the first rows; by finding, for each integer k in the new matrix, the smallest column number, for P, and smallest row number, for Q, where there is an appearance of k. This process is repeated until the new ball matrix has no entries. In example 5.1,

$$P(B) = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \\ 2 & 3 \end{pmatrix} \quad \text{and}$$

$$Q(B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 3 & & & & & & & \end{pmatrix}.$$

It still needs to be shown that constructing the tableaux by the matrix ball method coincides with constructing them by insertion. For this we will refer to [1].

Bibliography

[1] William Fulton. *Young Tableaux*. London Math. Society, 1935.

[2] E. Ruch and I. Gutman. The branching extent of graphs. *J. Combin. Inform. System Sci.*, 4:285 – 295, 1979.