1

Lecture 1

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \; x^n = \sum h_n x^n$$

So $\frac{1}{1-x}$ is the generating function for $1, 1, \ldots 1 = h_n, \ldots$

Ex The number of non-negative solutions to $e_1 + \cdots + e_k = n$ is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$

The generating function for the number of n solutions to $e_1 + \cdots + e_k = n$ is

$$\sum_n \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k}$$

Ex The number of different boxes of size n of 4 kinds of donuts in unlimited supply is

$$\sum \binom{n+4-1}{4-1} x^n = \frac{1}{(1-x)^4}$$

We have seen many variations of this problem (see page 171)

Here we look at more. What is the generating function for the number of donuts if we want an even number of sugar, an odd number of chocolate and at ~~least~~ most 5 plain?

$$(1 + x^2 + x^4 + \cdots)(x + x^3 + \cdots)(1 + x + \cdots + x^5)$$

$$= \frac{1}{1-x^2} \, \frac{x}{1-x^2} \, \frac{1-x^6}{1-x}$$

Ex Find the number of non-negative solutions to

$$3x_1 + 5x_2 + x_3 = n \qquad n = 0, 1, \ldots$$

x_1 is in mult of 3 : $1 + x^3 + x^6 + \cdots$

x_2 is in mult of 5 $\quad 1 + x^5 + x^{10} + \cdots$

x_3 is in mult of 1 $\quad 1 + x + \cdots$

Product $= \dfrac{1}{1-x^3} \dfrac{1}{1-x^5} \dfrac{1}{1-x} = g(x)$

Problems page 259-260 $\quad$ 13, 14, 16, 17, 18, 19

Some solutions to problems page 259-260

1a Let $f_0, \ldots f_n$, be the Fibonacci sequence
Find a closed value for

$$f_1 + f_3 + f_5 + \quad + f_{2n-1}$$

$f_1 = 1$

$f_1 + f_3 = 1 + 2 = 3 \qquad f_4$

$f_1 + f_3 + f_5 = 1 + 2 + 5 = 8 \qquad f_6$

$f_1 + f_3 + f_5 + f_7 = 8 + 13 = 21 \qquad f_8$

Conjecture $f_1 + \quad + f_{2n-1} = f_{2n} \quad *$

Induction $f_1 = 1 = f_2$

Assume $*$

$f_1 + \quad + f_{2n-1} + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2}$

(3)

3b $3 \mid f_n \iff 4 \mid n$

$f_1 = 1 \bmod 3$ $\quad$ $f_5 = 2 \bmod 3$

$f_2 = 1 \bmod 3$ $\quad$ $f_6 = 2 \bmod 3$

$f_3 = 2 \bmod 3$ $\quad$ $f_7 = 1 \bmod 3$

$f_4 = 0 \bmod 3$ $\quad$ $f_8 = 0 \bmod 3$

Use induction. Beginning steps are above.

Assume result for f_j $j \le 4n$

$f_{4n} = 0 \bmod 3$ assumption

$f_{4n-1} = 1 \bmod 3$ or $f_{4n-1} = 2 \bmod 3$

$f_{4n+1} = 1 \bmod 3$ $\quad$ $f_{4n+1} = 2 \bmod 3$

$f_{4n+2} = 1 \bmod 3$ $\quad$ $f_{4n+2} = 2 \bmod 3$

$f_{4n+3} = 2 \bmod 3$ $\quad$ $f_{4n+3} = 1 \bmod 3$

$f_{4n+4} = 0 \bmod 3$ $\quad$ $f_{4n+4} = 0 \bmod 3$

(Add the 2 previous values to get the new one)

It all works so result holds

4

(4) Generating function solutions to recurrence relations. Show by example

Solve $h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0$
with $h_0 = 0 \quad h_1 = 1 \quad h_2 = -1$

Let $g(x) = h_0 + h_1 x + h_2 x^2 + \ldots$
be the generating function.
Then

$$g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3$$
$$x\,g(x) = \qquad h_0 x + h_1 x^2 + h_2 x^3 + \ldots$$
$$-16x^2 g(x) = \qquad -16h_0 x^2 - 16h_1 x^3 \ldots$$
$$+20x^3 g(x) \qquad 20h_0 x^3 + \ldots$$

$\rightarrow$

$$g(x) + x\,g(x) - 16x^2 g(x) + 20x^3 g(x) =$$
$$(1 + x - 16x^2 + 20x^3)\,g(x) =$$
$$h_0 + (h_1 + h_0)x + (h_2 + h_1 - 16h_0)x^2 + (h_3 + h_2 - 16h_1 + 20h_0)x^3 + \ldots$$
$$(h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3})x^n + \ldots$$

Note all terms after x^2 have the recurrence relation in them so are 0
Also $h_0 = 0 \quad h_1 = 1 \quad h_2 = -1 \longrightarrow$

$$(1 + x - 16x^2 + 20x^3)\,g(x) = 0 + (1+0)x + (-1+1-0)x^2$$
$$= x$$

$$g(x) = \frac{x}{1 + x - 16x^2 + 20x^3}$$

$$1 + x - 16x^2 + 20x^3 = (1-2x)^2(1+5x)$$

5)

We use partial fractions

$$\frac{x}{1+x-16x^2+20x^3} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+5x}$$

Clear of fractions

$$x = (1-2x)(1+5x)A + (1+5x)B + (1-2x)^2 C$$

To find A, B, C:

$x = \frac{1}{2}$ $\quad \frac{1}{2} = B\left(\frac{7}{2}\right) \qquad B = 1/7$

$x = -\frac{1}{5}$ $\quad -\frac{1}{5} = C\left(1+\frac{2}{5}\right)^2 = C\left(\frac{7}{5}\right)^2$

$$C = -5/49$$

$\rightarrow$ $\quad A = -2/49$

$$g(x) = -\left(\frac{2}{49}\right)\frac{1}{1-2x} + \left(\frac{7}{49}\right)\frac{1}{(1-2x)^2} - \frac{5}{49}\frac{1}{(1+5x)}$$

$$\frac{1}{1-2x} = \sum 2^n x^n$$

$$\frac{1}{1+5x} = \sum (-5)^n x^n$$

$$\frac{1}{(1-2x)^2} = \sum \binom{n+1}{n} 2^n x^n = \sum (n+1) 2^n x^n$$

$\rightarrow$

$$g(x) = -\frac{2}{49}\sum 2^n x^n + \frac{7}{49}\sum (n+1)2^n x^n - \frac{5}{49}\sum (-5)^n x^n$$

$\rightarrow$

$$h_n = -\frac{2}{49} 2^n + \frac{7}{49}(n+1)2^n - \frac{5}{49}(-5)^n$$

Problems p 263 40, 48 (a, c, e)

CSC-MA 416

Lesson II

March 17, 2020

Recurrence relations

A system $h_n=a_1h_{n-1}+a_2h_{n-2}+...a_kh_{n-k}+b_n$, $n\geq k$ is called a linear recurence relation. The as are fixed but need not be constant. We would like to get $h_n=$ a function, said to be in closed form.

Example Let $h_n=2h_{n-1}+1$ where $h_0=1$. Giving h_0 is called an intial condition. In the general problem of the last paragraph, one would give k initial conditions. The one could recursively solve for larger and larger values of n. Here h_1-3, $h_2=7$, $h_3=15$. We notice a pattern, $h_n=2^{n+1}$-1. We prove this is the correct form using induction. The first cases hold. So suppose $h_n=2^{n+1}$-1. Then $h_{n+1}=2h_n+1=2(2^{n+1}\text{-}1)+1=2^{n+2}$-1. The result holds. We solved this by finding a pattern and then showed the result by induction. This is a good attack. We will see several other attacks in the this section. See page 262 number 38 for more of this type of problem.

To introduce another attack, we begin with a familiar example. Let f_n be the n'th term in the Fibonacci sequence. This is given by $f_n=f_{n-1}+f_{n-2}$ where $f_0=0$ and $f_1=1$. This is a linear recurrence relation with constant coefficients since the coefficients in front of the terms are all constant (value =1 here). It is linear because each f is raised to the first power. It is of degree 2 since there are three terms. In the problems that we will consider we quess at a solution $f_n=q^n$ where q is not 0. Substitute the guess into the equation to get $q^n\text{-}q^{n-1}\text{-}q^{n-2}=0$. Since $q\neq 0$, factor it out and cancel to get q^2-q-1=0. Solving the quadratic gives $q=(1+\sqrt{(5)})/2$ and $q=(1-\sqrt{(5)})/2$. As we noted, any linear combination is also a solution $f_n=c_1((1-\sqrt{(5)})/2)^n+c_2((1-\sqrt{(5)})/2)^n$. To find c_1 and c_2 we let n=0 and n=1 into this equation and use the initial conditions.See page 210 and 211 for details.

Solving the recurrence in the last paragraph is an example of the process we now describe. Given recurrence $h_n=a_1h_{n-1}+\ a_2h_{n-2}\ \dots\ +\ a_kh_{n-k}$, rewrite as $h_n\text{-}a_1h_{n-1}$ - a_2h_{n-2} ... $-a_kh_{n-k}=0$ Let $h_k=q^k$ and substitute into the last equation, factor out q^{n-k} to get a polynomial of degree k. We then need to solve for the roots, this could be a very hard problem. But if we can solve for the roots, $r_1...r_k$, then (ASSUMING THAT THE ROOTS

ARE ALL DIFFERENT), $h_n=c_1q^{r_1}...+c_kq^{r_k}$. To Find the coefficients, we use the initial conditions. The process is illustrated by the discussion of Fibonacci on pages 210-212. The verification of this is shown on pages 230 and 232. Use is made of the Vandermonde matrix for the roots and needs to have an inverse which is true if and only if the roots are distinct. The whole idea parallels one for linear differential equations.

Example. A familiar recurrence relation is one for derangements $D_n=(n-1)D_{n-1}+(n-1)D_{n-2}$. The coefficients are not linear, so the above method to find D_n will not work.

Example Solve $h_n=h_{n-1}+9h_{n-2}-9h_{n-3}$. This becomes $q^n-q^{n-1}-9q^{n-2}+9q^{n-3}=0$. By trying the possible rational roots, we find that they are 1,3 and -3. Therefore $h_n=c_1 1^n+c_2 3^n+c_3(-3)^n$ Initial conditions are given as $h_0=0$, $h_1=1$ and $h_2=2$. These give equations

$0=c_1+c_2+c_3$

$1=c_1+c_2 3+c_3(-3)$

$2=c_1+c_2 9+c_3 9$

Eliminating gives

$2c_2-4c_3=1$

$8c_2+8c_3=2$

Then

$c_2=1/3$ and $c_3=-1/12$

Then $c_1=-1/4$.

Hence $h_n=-1/4+(1/3)(3^n)-(1/12)(-3)^n$.

Example Let h_n be the number of ways that a 1 by n cheesboard can be colored by red, blue and white moninmoes where two reds can not appear next to each other. Clearly $h_1=3$, $h_2=4$. Set $h_0=1$. If a blue or white occurs first, then the coloring can occur in h_{n-1} ways. if a red occurs first, then the second tile must be blue or white. After that the coloring proceeds in h_{n-2} ways. Thus the recursion is $h_n=2h_{n-1}+2h_{n-2}$. We obtain the equation $x^n-2x^{n-1}-2x^{n-2}=0$. This reduces to $x^2-2x-2=0$. The quadratic formula gives $x=1+\sqrt(3)$ and $x=1-\sqrt(3)$. So $h_n=c_1(1+\sqrt(3))^n+c_2(1-\sqrt(3)^n$

The intial conditions n=0 and 1 give

$c_1+c_2=1$

$c_1(1+\sqrt(3))+c_2(1-\sqrt(3))=3$

Solving, we find $c_1=(2+\sqrt(3))/2\sqrt(3)$, $c_2=(-2+\sqrt(3))/2\sqrt(3)$

Then $h_n=[(2+\sqrt(3))/2\sqrt(3)](1+\sqrt(3))^n+[(-2+\sqrt(3))/2\sqrt(3)](1-\sqrt(3)^n$

REPEATED ROOTS.

Example: Find the solution to $h_n=4h_{n-1}-4h_{n-2}$.

This reduces to $x^2-4x+4=0$ with roots 2,2. The previous method gives just a linear combination of just one solution, not enough, we need 2. Instead use 2^n and $n2^n$, so $h_n=c_1 2^n + c_2 n2^n$ is the general solution. As usual, intial conditions will determine the constants.

If the root, say r, is repeated k times, we use linear combinations of $h_n=r^n$, nr^n, n^2r^n,...$n^{k-1}r^n$
* $h_n=c_1r^n+c_2\, nr^n + c_3n^2r^n,\ldots c_kn^{k-1}r^n$

The more general case if a number of roots occur in the problem, each one gets a solution * and then take sums of all the *'s

Example. Find the general solution to $h_n=-h_{n-1}+3h_{n-2}+5h_{n-3}+2h_{n-4}$. The usual procedure leaves us with $x^4+x^3-3x^2-5x-2=0$. Doing the rational root test finds roots -1,-1,-1,2. There are two types of roots, we need to follow the procedure in the last example twice, once for each type of root. It is straightforward and we only need to write down the answer

$h_n = c_1(-1)^n + c_2n(-1)^n + c_3n^2(-1)^n+c_4 2^n$

Given initial conditions $h_0=1$, $h_1=0$, $h_2=1$ and $h_3=2$ from which we can find the particular constants, we obtain the equations

$c_1 + c_4=1$

$-c_1 - c_2 - c_3 + 2c_4=0$

$c_1 + 2c_2 + 4c_3 + 4c_4=1$

$-c_1 - 3c_2 - 9c_3 + 8c_4=2$

Solving this system gives $c_1 = 7/9$, $c_2=-3/9$, $c_3=0$ and $c_4=2/9$.

Hence $h_n=7/9(-1)^n-3/9n(-1)^n+2/9(2)^n$

Problems page 261: 31, 34, 35

There is a method using generating functions that solve recurence relations. We show this by example.

Solve the recurrence $h_n+h_{n-1}-16h_{n-2}+20h_{n-3}=0$ with initial conditions $h_0=0$, $h_1 = 1$, $h_2=-1$.

Let $g(x)=h_0+h_1x+h_2x^2+..$, the generating function for the solution. Then

$g(x)+xg(x)-16x^2g(x)+20x^3g(x)$ [Notice the coefficients come from the recursion relation] gives

$(1+x-16x^2+20x^3)g(x)= h_0+(h_1+h_0)x+(h_2+h_1-16h_0)x^2+(h_3+h_2-16h_1+20h_0)x^3+\ldots(h_n+h_{n-1}-16h_{n-2}+20h_{n-3})x^n$

This last coefficient is 0 for $n \geq 3$ from the recurrence relation. Using the initial conditions, this large expression becomes $(1+x-16x^2+20x^3)g(x)=x$. and $g(x)=x/(1+x-16x^2+20x^3)$. We are going to use partial fraction techniques.

$1+x-16x^2+20x^3=(1-2x)^2)(1+5x)$. We work with

$x/(1+x-16x^2+20x^3)=a/(1-2x)+b/(1-2x)^2+c/(1+5x)$
Clear of fractions
$x=(1-2x)(1+5x)a+(1+5x)b+(1-2x)^2c$
Equating like powers gives.
$a+b+c=0$
$3a+5b-4c=1$
$-10a+4c=0$
From which we find
$a=-2/49$, $b=7/49$ and $c=-5/49$
Hence
** $g(x)=x/(1+x-16x^2+20x^3)=(2/49)/(1-2x)+(7/49)/(1-2x)^2-(5/49)/(1+5x)$
We know the expansions for $1/(1-cx)^k$ from page 220. Using this information
$1/(1-2x)=\sum_{k=0}^{\infty} 2^k x^k$
$1/(1-2x)^2=\sum_{k=0}^{\infty} \binom{k+1}{k} 2^k x^k=\sum_{k=0}^{\infty} (k+1)2^k x^k$
$1/(1+5x)=\sum_{k=0}^{\infty}(-5)^k x^k$.
Substitute this into the right side of ** to get
$g(x)=-2/49\sum_{k=0}^{\infty} 2^k x^k+7/49\sum_{k=0}^{\infty} (k+1)2^k x^k-5/49\sum_{k=0}^{\infty}(-5)^k x^k=\sum_{k=0}^{\infty}[-(2/49)\,2^k+(7/49)(k+1)2^k-(5/49)(-5)^k]x^k$
Thus $h_k=\sum_{k=0}^{\infty}[-(2/49)\,2^k+(7/49)(k+1)2^k-(5/49)(-5)^k]$

Problems Page 263 40, 48 a,c,e

Lecture 3

Non homogeneous Equations

$$h_n = a_1 h_{n-1} + \cdots + a_k h_{n-k} + b_n \quad , b_n \neq 0$$

Ex $h_n = 2h_{n-1} + 1 \quad , \quad h_0 = 0$

$h_1 = 0 + 1 = 1$

$h_2 = 2 + 1 = 3$

$h_3 = 6 + 1 = 7$

$h_4 = 14 + 1 = 15$ Guess $h_n = 2^n - 1$ *

Induction. Assume *

$h_{n+1} = 2h_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1$ ✓

Ex $h_n = 3h_{n-1} - 4n \quad h_0 = 2$

$h_n - 3h_{n-1} = -4n$

① Solve homogeous part

$h_n - 3h_{n-1} = 0 \qquad h_n = x^n$

$x^n - 3x^{n-1} = 0 \rightarrow x - 3 = 0 \qquad x = 3$

$h_n = 3^n$

Guess at particular solution

$h_n^p = rn + s$ (since $b_n = -4n$ is a polynomial of degree 1)

→ $rn + s = 3(r(n-1) + s) - 4n$

Read off coeff. of n and 1

$r = 3r - 4 \quad , \quad 4 = 2r \quad , \quad r = 2$

$s = -3r + 3s \quad , \quad 2s = 3r \quad , \quad s = 3$

$h_n^p = 2n+3$

$\rightarrow h_n = C3^n + 2n+3$

$2 = h_0 = C+3$

$C = -1$

$h_n = -3^n + 2n+3$

Ex $\quad h_n = 2h_{n-1} + 3^n$

As before, homogeneous solution

$h_n = 2h_{n-1}$

$x^n = 2x^{n-1} \qquad x = 2$

$h_n = 2^n$

For $b_n = 3^n$, let $h_n = d\,3^n$

$d\,3^n = 2d\,3^{n-1} + 3^n$

$3d = 2d + 3$

$d = 3$

$h_n = C2^n + 3\cdot 3^n$

Ex $\quad$ Towers of Hanoi

There are 3 pegs and n circular disks of increasing size on one peg, largest at the bottom.

The disks are to be moved to another peg, one disk at a time, always larger disks below smaller disks.

How many steps?

h_n = number of steps for n disks

$h_1 = 1 \qquad h_2 = 3$

Find a recursion.

Do the top n-1 first. Takes h_{n-1} steps
Move the last, largest disk
to 3rd peg. Then move the other $n-1$,
takes another h_{n-1} steps

$\therefore \quad h_n = 2h_{n-1} + 1 \quad h_0 = 0$

Can solve this as in last examples to get

$$h_n = 2^n + 1$$

Now we solve the problem with generating functions. Let

$$g(x) = \sum h_n x^n$$

$$g(x) = h_0 + h_1 x + \qquad + h_n x^n + \cdots$$

$$x g(x) = \qquad h_0 x \qquad + h_{n-1} x^n + \cdots$$

$$g(x) - 2x g(x) = h_0 + (h_1 - 2h_0)x + (h_2 - 2h_1)x^2 + \cdots$$

$$= 0 + 1x + x^2 + \cdots$$

since $h_n - 2h_{n-1} = 1$ and $h_0 = 0$

$\rightarrow$

$$(1-2x)g(x) = x + x^2 + \cdots$$

$$= \frac{x}{1-x}$$

$$g(x) = \frac{x}{(1-2x)(1-x)} = \frac{A}{1-x} + \frac{B}{1-2x}$$

$$x = A(1-2x) + B(1-x)$$

$x = 1 \qquad 1 = A(-1) \qquad A = -1$

$x = \frac{1}{2} \qquad \frac{1}{2} = B\frac{1}{2} \qquad B = 1$

$$g(x) = \frac{-1}{1-x} + \frac{1}{1-2x} = -\sum x^n + \sum 2^n x^n$$

$$\qquad\qquad\qquad\qquad\qquad \shortparallel$$

$$h_n = 2^n - 1 \longleftarrow \sum(-1+2^n)x^n$$

Solve $h_n = 3h_{n-1} + 3^n \quad h_0 = 2$
by generating functions

$g(x) = h_0 + h_1 x + \cdots$

$g(x) - 3xg(x) =$

$h_0 + (h_1 - 3h_0)x + (h_2 - 3h_1)x^2 + \cdots$

$= 2 + 3x + 3^2 x^2 + \cdots + 3^n x^n$

$= 1 + \dfrac{1}{1-3x}$

$\Rightarrow \quad (1-3x)g(x) = 1 + \dfrac{1}{1-3x}$

$$g(x) = \frac{1}{1-3x} + \frac{1}{(1-3x)^2}$$

$$= \sum 3^n x^n + \sum \binom{n+1}{n} 3^n x^n$$

$$= \sum \left(3^n + (n+1)3^n \right) x^n$$

$$h_n = 3^n + 3^n(n+1)$$
$$= 3^n(n+2)$$

Lecture 4

Catalan Numbers Ch. 8

$$C_n \equiv \frac{1}{n+1}\binom{2n}{n} \qquad n=0,1,2,\ldots$$

$C_0=1$ $C_1=1$ $C_2=2$ $C_3=\frac{1}{4}\binom{6}{3}=5$

$C_4=14$ $C_5=42$ $C_9=4862$

Thm. Consider $a_1, \ldots a_{2n}$ where there are n 1's and n -1's. $\ni$ partial sum $a_1 + + a_k$ is always ≥ 0
The number of these sequences (Fixed n)
is $\frac{1}{n+1}\binom{2n}{n} = C_n$

Ex The 8 term sequence has $n=4$, $C_n=14$

1 1 −1 −1 1 −1 1 −1 would work
1 1 −1 −1 −1 1 1 −1 would not

The sequence is acceptable if it works otherwise it is unacceptable. Let A_n be be the number of acceptable sequences and U_n be the number of nonacceptable. Remember there are $2n$ terms in these sequences. The total number of these sequence, both accep and unacceptable is $\binom{2n}{n}=\frac{(2n)!}{n!\,n!}$ (We are choosing n out of $2n$ for the 1's.

$\therefore$ $A_n + U_n = \binom{2n}{n}$

A_n = number of acceptable
U_n = number of unacceptable

2

We compute U_n. If $a_1, \ldots a_{2n}$ is in U_n $\exists$ a first k $\ni$ $a_1 + \ldots + a_k < 0$. Then $a_1 + \ldots + a_{k-1} = 0$ and $a_k = -1$. So k is odd. Now reverse signs of each of the first k terms, leave rest alone

i.e. 1 1 -1 -1 -1 1 1 -1 becomes
-1 -1 1 1 1 1 1 -1

We have $n+1$ 1's $n-1$ -1's

The process is reversable. Given the second type of sequence $\exists$ first place where the number of +1 exceeds the number of -1 (Step 5 in the example). Change signs up to that point -1 -1 1 1 1 1 1 -1
$\longrightarrow$ 1 1 -1 -1 -1 1 1 -1

This is in U_n. The number of sequences of $n+1$ +1's and $n-1$ -1's is $\binom{2n}{n+1}$ (Choose $n+1$ places for the 1's from $2n$.

$$U_n = \frac{(2n)!}{(n+1)!\,(n-1)!}$$

$$\to \quad A_n = \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n+1)!\,(n-1)!}$$

$$= 2n! \, \frac{1}{n!\,(n-1)!} \left[-\frac{1}{n+1} + \frac{1}{n} \right]$$

$$= \frac{2n!}{n!\,(n-1)!} \, \frac{1}{n(n+1)} = \frac{1}{n+1} \binom{2n}{n}$$

Many problems can be cast in the ideas of the last theorem.

Ex A person walks n blocks N and n blocks E to work. They always stay N of the diagonal (or on it). How many paths are there

Identifying N with a +1 and E with −1. Each block ~~adds~~ adjoins a +1 or −1 to the list. The list must always ~~stay~~ stay ≥ 0. This is the last Thm. So answer is $\binom{2n}{n}\frac{1}{n+1} = C_n$

Ex There are $2n$ people in line at a movie that costs 50 ~~cents~~ to enter (This is an old problem!) n people have 50¢, n people have \$1. The movie starts with no change. How many ways can the people line up so that everyone can get in

① Not distinguishing people, just how much money they have we give 50¢ people a +1 and \$1 people a −1. The +1 is always must be ≥ 0 −1. So this is the old problem: Ans C_n

② If we distinguish the people, $\exists$ $n!$ ways to arrange the 50¢ people, $n!$ ways to arrange the \$ people. Total

$$n!\,n!\,C_n = \frac{n!\,n!}{(n+1)!}\binom{2n}{n}$$

4

The C_n satisfy a recurrence relation,
it is homogeneous but coeff are not constants
It is of order 1.

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{(2n)!}{n!\,n!}$$

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} = \frac{1}{n}\frac{(2n-2)!}{(n-1)!\,(n-1)!}$$

$$\frac{C_n}{C_{n-1}} = \frac{\frac{1}{n+1}\frac{(2n)!}{n!\,n!}}{\frac{1}{n}\frac{(2n-2)!}{(n-1)!\,(n-1)!}} = \frac{n}{n+1}\frac{(2n)(2n-1)}{n\,n} = \frac{4n-2}{n+1}$$

$$\frac{C_n}{C_{n-1}} = \frac{4n-2}{n+1} \rightarrow C_n = \frac{4n-2}{n+1} C_{n-1}$$

$$C_0 = 1$$

Ex $\quad C_5 = \frac{20-2}{6}\, 14 = \frac{18 \cdot 14}{6} = 6 \cdot 7 = 42$

Pseudo Catalan Numbers

Let

$$C_n^* = n!\, C_{n-1}$$

$C_1^* = 1 \quad C_0 = 1$ Catalan recurrence

$$C_n^* = n!\, C_{n-1} = n!\, \frac{4n-6}{n}\, C_{n-2}$$

$$= (4n-6)(n-1)!\, C_{n-2}$$

$$= (4n-6)\, C_{n-1}^*$$

$\rightarrow$

$C_n^* = (4n-6)\, C_{n-1}^*$ is a recurrence for the Pseudo Catalan numbers

$C_1^* = 1$

$\rightarrow \quad C_1^* = 1 \quad C_2^* = 2 \quad C_3^* = 12 \quad C_4^* = 120$

i.e. $\quad C_4^* = (16-6)\, C_3^* = 10 \cdot 12 = 120$

$C_5^* = 1680 \quad C_6^* = 30240$

Note $\quad C_n^* = n!\, C_{n-1} = n!\, \frac{1}{n} \binom{2n-2}{n-1}$

$$= (n-1)! \binom{2n-2}{n-1}$$

Ex How many ways can we multiply $a_1, \ldots, a_n$

Let h_n be the number

$h_1 = 1 \quad h_2 = 2 \quad a_1 a_2 \quad a_2 a_1$

$h_3 = 12$

$a_1(a_2 a_3) \quad a_2(a_1 a_3) \quad a_3(a_1 a_2)$

$a_1(a_3 a_2) \quad a_3(a_2 a_1) \quad a_2(a_1 a_3)$

$(a_2 a_3) a_1 \quad (a_3 a_1) a_2 \quad (a_1 a_2) a_3$

$(a_3 a_2) a_1 \quad (a_2 a_1) a_3 \quad (a_1 a_3) a_2$

6 We look for a recurrence relation
If we have $n-1$ elements, h_{n-1} is the number of ways to align them. How much does this increase when we insert an nth element?

Ex $\quad$ * $(a_1 a_2)[a_6 (a_3 a_4) a_5] \quad n-1=5$

To insert a_6 into the $[(a_3 a_4) a_5]$:

a_6 can go on either side of $a_3 a_4$

$(a_6 (a_3 a_4)) a_5$

$((a_3 a_4) a_6) a_5$

or with the a_5

$(a_3 a_4)(a_6 a_5)$

$(a_3 a_4)(a_5 a_6)$

So for each of the $n-2$ ways of writing the elements $\exists$ 4 ways to insert a_n

$4(n-2)$ terms

If we insert a_6 on the outside, $\exists$ 2 ways (left or right)

So each term, as in *, has

$4(n-2)+2$ ways to put in the new term

$\therefore \quad h_n = (4n-6)h_{n-1}$ is the recursion

$h_1 = 1$

This is exactly the recursion + initial conditions for C_n^* i.e.

$$h_n = (n-1)!\binom{2n-2}{n-1} = C_n^*$$

So

$h_1 = 1 \quad h_2 = 2 \quad h_3 = 12 \quad h_4 = 120$

$h_5 = 1680 \quad h_6 = 30240$