Lesson 11

(1)

We will look at problems today

1. Solve the recursion

$h_n = 3h_{n-1} - 4, \quad h_0 = 3$

using generating functions

Let $g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots$

$-3x\,g(x) = \quad -3h_0 x \quad -3h_1 x^2 \quad -3h_2 x^3 - \cdots$

$(1-3x)g(x) = h_0 + (h_1 - 3h_0)x + (h_2 - 3h_1)x^2 \cdots$

$= 3 \quad -4x \quad -4x^2 \cdots = 3 - 4x(1 + x + x^2 + \cdots)$

$$g(x) = \frac{3}{1-3x} + \frac{-4x}{(1-3x)(1-x)}$$

$$\frac{-4x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

$-4x = A(1-x) + B(1-3x)$

$0 = A + B \qquad A = -B$

$-4 = -A - 3B = B - 3B = -2B \qquad B = 2 \qquad A = -2$

$$g(x) = \frac{3}{1-3x} + \frac{-2}{1-3x} + \frac{2}{1-x}$$

$= 3\sum 3^n x^n - 2\sum 3^n x^n + 2\sum x^n$

$h_n = 3^{n+1} - 2\,3^n + 2$

$= 3^n + 2$

2 $2n$ points are on a circle. Show that the number of ways to join these points in pairs by line segments so that the segments never intersect is C_n

Let g_n be the number of ways with $2n$ points

$n=1$  $g_1 = 1$

$n=2$  $g_2 = 2$

Verify that $g_3 = 5$

Fix a point at the top of the circle. Line to other points can only go to points that are an odd number from the top point since the points on either side of the segment must be joined without intersecting the first line. If there are $2k$ points on the left of the line, there are $2n-2k-2$ points on the right since 2 points are used by the line. The on either side of the line, the same problem presents itself, connecting points without lines crossing. By induction, on the left

③ There are g_k ways to achieve the goal, on the right, there are g_{n-k-1} ways. The total is then

$$g_k \, g_{n-k-1}$$

This starts when $k=0$ (the point connecting to A is next to A) and goes to $n-1$

Hence $g_n = g_0 g_{n-1} + g_1 g_{n-2} + \cdots + g_{n-1} g_0$

$g_1 = 1 \quad g_2 = 2 \qquad$ define $g_0 = 1$

This answer (and method) is alot like the Triangulation problem on page 254 where the answer had the form

$$h_1 h_{n-1} + \cdots + h_{n-1} h_1 = h_n$$

$$h_1 = 1 \quad h_2 = 1$$

With some complicated calculations it was found ~~this~~ $h_n = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1}$

g_n has another term but the same sequence of numbers, so

$$g_n = \frac{1}{n+1}\binom{2n}{n} = C_n$$

This is a great work saver, the observation that our problem fit an old solved problem allowed us to use the work

Note $g_n = h_{n+1}$ Check a few small n in the sums for g_n and h_n using initial conditions

3\. Prove the Fibonnaci Sequence is a solution to

$$a_n = 5\,a_{n-4} + 3\,a_{n-5} \quad \text{where}$$

$$a_0 = 1 \quad a_1 = 1 \quad a_2 = 1 \quad a_3 = 2 \quad a_4 = 3 \quad a_5 = 5$$

Recall $\quad f_0 = 1 \quad f_1 = 1 \quad f_2 = 1 \quad f_3 = 2 \quad f_4 = 3 \quad f_5 = 5$

$$f_n = f_{n-1} + f_{n-2} = f_{n-2} + f_{n-3} + f_{n-3} + f_{n-4} =$$

$$f_{n-3} + f_{n-4} + f_{n-4} + f_{n-5} + f_{n-4} + f_{n-5} + f_{n-4}$$

$$= f_{n-4} + f_{n-5} + f_{n-4} + f_{n-4} + f_{n-5} + f_{n-4} + f_{n-5} + f_{n-4}$$

$$= 5\,f_{n-4} + 3\,f_{n-5}$$

So $a_n = f_n$ for all (Since the initial conditions are the same

$$f_n = 5\,f_{n-4} + 3\,f_{n-5}$$

* 5 divides $f_n \longleftrightarrow$ 5 divides RHS

$\longleftrightarrow 5 \mid 3 f_{n-5} \longleftrightarrow 5 \mid f_{n-5}$

For $n = 1, 2, 3, 4, 5$ $\quad 5 \mid n \leftrightarrow 5 \mid f_n \quad (f_5 = 5)$

So from * and $5 \mid n \leftrightarrow 5 \mid f_n \quad n = 1, 2, 3, 4, 5$

we get $5 \mid f_n$ iff $5 \mid n$

4. Using pennies, nickles, dimes and quarters how many ways can we make change adding up to n¢?

e_1 = number of pennies e_2 = number of nickles
e_3 = number of dimes e_4 = number of quarters

$$e_1 + 5e_2 + 10e_3 + 25e_4 = n$$

We get generating function

$$(1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots)(1+x^{10}+\cdots)(1+x^{25}+x^{50}+\cdots)$$

$$= \frac{1}{1-x}\,\frac{1}{1-x^5}\,\frac{1}{1-x^{10}}\,\frac{1}{1-x^{25}}$$

5 Let $h_0, h_1, \ldots h_n$ be sequence $h_n = \binom{n}{2}$
Find the generating function

Note $\binom{n}{2} = \binom{n}{n-2}$

Recall That $\frac{1}{(1-x)^k} = \sum \binom{n+k-1}{n} x^n$

If $k=3$, $\frac{1}{(1-x)^3} = \sum \binom{n+2}{n} x^n$

The coefficents are off by x^2, so

$$\frac{x^2}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{n} x^{n+2}$$

or

$$= \sum_{n=2}^{\infty} \binom{n}{n-2} x^n$$

~~LIST OF FIGURES~~

Dear Class

The next sheet is a Problem set on pre graph work. It is to hand in by April 2

Homework 6

I Determine the generating function for the number of nonnegative integual solutions to

$$3e_1 + 5e_2 + e_3 + 2e_4 = n$$

II Solve the recurrence relation using generating functions

$$h_n = 4h_{n-1} - 4h_n \quad h_0 = 0 \quad h_1 = 1$$

III Solve the recurrence relation using generating functions

$$h_n = 2h_{n-1} + 3 \quad h_0 = 1$$

IV Prove the following about Fibonnaci numbers

f_n is divisible by 4 if and only if n is divisible by 6

V Prove that thee number of 2 by n matricies

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \end{pmatrix}$$ made up from numbers $1, 2, \ldots, 2n$

such that
$$x_{11} < x_{12} < \cdots < x_{1n}$$
$$x_{21} < x_{22} < \quad < x_{2n}$$
$$x_{11} < x_{21}, \; x_{12} < x_{22}, \ldots \; x_{1n} < x_{2n}$$

equals C_n, the n Catalen number

HinT: This is done as Examples on page 268