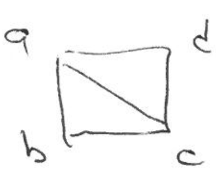
HAMILTON CYCLES

Def A cycle is a closed walk where no vertex is repeated except the first equals the last

If every vertex is included it is called a Hamilton cycle

If we remove the closed condition the graph is called a Hamilton path. So all vertices are included exactly once



a – c – d – a cycle

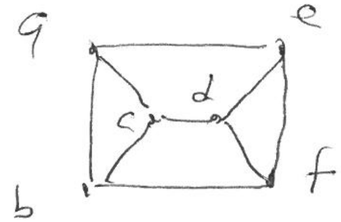
a – b – c – d – a Hamilton cycle

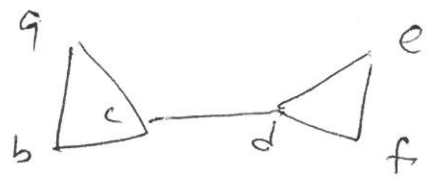
b – c – a – d Hamilton path

If $|G| = n$, a Hamilton cycle has n edges. A Hamilton path has $n-1$ edges

Ex Every complete graph K_n has a Hamilton cycle, numerous ones since every vertex is connected by an edge to every other vertex

The following example display ideas we will persue





Hamilton cycle

a-b-c-d-f-e-a

There is no cycle because the graph partions into 2 subgraph connected by one edge

In the second graph, if we start on the left we would need to go through d twice, so it would not be a cycle. c-d is called a bridge. By definition there can only be one bridge between the two subgraphs Otherwise we do not call it a bridge. There is no Eulerien Trail either

The study of Hamilton cycles is a complex one. Unlike Eulerian Trails, there ~~x~~ is no necessary and sufficient condition for the to exist. A simple necessary condition is that a bridge can not exist. This follows for the same reason as in our example

We now look at a sufficient condition

Def: Let $|G| = n$. If every pair of (x and y) distinct vertices that are not adjacent has the property

$$\deg(x) + \deg(y) \geq n$$

then G is said to satisfy the Ore property

The Ore property is a sufficient condition for the existence of a

Hamilton cycle.

Proposition Let $|G| = n \geq 3$. If G satisfies the Ore condition the G is connected

Proof If G is not connected, it has subgraph H and K with no edges between verticies Suppose $|H| = h$ $|K| = k \to n = h+k$

For any $x \in H$, $y \in K$ $deg(x) < h$, $deg(y) < k$. Therefore $deg(x) + deg(y) < n$ Since x and y can not be adjacent (no edge between them), this contradicts the Ore property

Theorem. 1G123. If G satisfies the Ore condition, then G has a Hamilton cycle

Proof. Start with any vertex and form a path in both directions until we can not go further.

$$\gamma: y_1 - y_2 \qquad - y_m$$

So both y_1 and y_m are adjacent to only vertices on this path, otherwise we could go further. The idea is that we have built this path one step at a time. There could be longer paths if we made different vertex choices in each step.

There are 2 possibilites

y_1 and y_m are adjacent or
y_1 and y_m are not adjacent

In this algorithm we can run into each case repectly until we are done constructing the Hamilton cycle

I y_1 and y_m are adjacent

If $m = n$, done and this is where we will always finally end up

If $m \neq n$: Since the graph is connected, there exists z not on γ and y_k on γ such that z and y_k are adjacent

Reorder γ with z included by

$$z - y_k - y_{k+1} \cdots - y_m - y_1 - \cdots - y_{k-1}$$

We used y_m and y_1 are adjacent

We now have diffent vertices at either end and go back and try to add new vertices (in either direction)

II y_1 and y_m are not adjacent

Let $\deg(y_1) = r$, $\deg(y_m) = s$

Then $r + s \geq n$ by the Ore cond.

Since we can not extend further, y_1 and y_m are only adjacent to vertices in the present path, call it γ (again) Claim

Each of the r verticies adjacent to y_1 is proceeded by a vertex (of course) AND at least one of them is adjacent to y_m. Call this vertex y_k and its predecessor y_{k-1}. So $y_1 - y_k$ and $y_{k-1} - y_m$

For if not, y_m is adjacent to at most $\underset{\text{vertices on } \gamma}{(m-1)} - \underset{\text{positions in front of vertices adjacent to } y_1}{r}$

$\rightarrow s \leq (m-1) - r$ or

$r + s \leq m - 1 \leq n - 1 \rightarrow \leftarrow$ Ore property

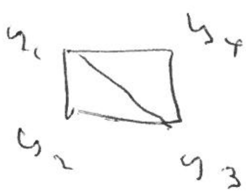
Form new path

$\gamma \quad y_1 - y_2 \quad y_{k-1} - y_m - y_{m-1} - \cdots - y_k$

This new γ has end vertices adjacent and we go back to search for making path longer as in I

Eventually, after adding vertices we must get to all n of them with final ones adjacent.

Examples

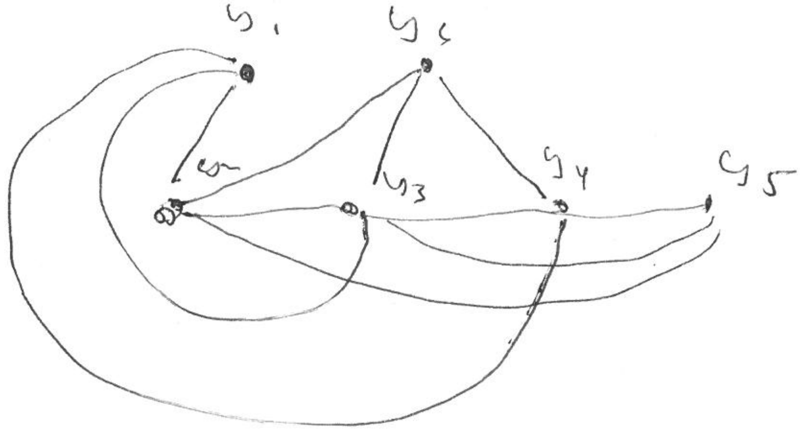


$\gamma: \; y_1 - y_3 - y_4$

y_1 is ~~not~~ adjacent to y_4

This is CASE I

$y_2 - y_3 \rightarrow y_2 - y_3 - y_4 - y_1$ is longer path



$n = 6$ All degrees ≥ 3

The Ore condition holds

Suppose $\gamma_i: y_1 \to y_2 - y_3 - y_4 - y_5$

$y_1 - y_4$, $y_5 - y_3$

So as in $\underline{\overline{II}}$

$y_1 - y_2 - y_3 - y_5 - y_4$

y_1 and y_4 are adjacent

CAN YOU THINK OF A GRAPH

which has

1 Both a HamiTon cycle and
a Eulerian Trail

2. A HamiTon cycle but
not a Eulerian Trail

3 A Eulerian Trail but
not a ~~Hamilton~~ Hamilton
cycle

4 Neither

?

Is the One condition necessary
for a HamilTon cycle?

Homework 2, 3, 4, 5, 10, 13, 14
20, 21, 29, 40, 49