Lesson 7

GRAPHS

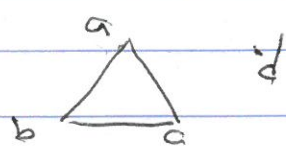
A graph G consists of a finite set, V, of elements called vertices and a set of pairs of elements from V, called edges, and denoted by E. We write $G = (V, E)$. The number of vertices is called the order of G.

If $(x, y) \in E$, then the edge $e = (x, y)$ is said to join x and y. Then x and y are called adjacent. Also x and e are said to be incident, as is y and e.

G can be given as a figure in the plane. Each vertex gets a point and each edge gets a curve joining x and y

Ex $V = \{a, b, c, d\}$

$E = \{(a,b), (a,c), (b,c)\}$



In the definition, only one edge can join vertices. In a different case when multiple edges can join the same vertices, the result is called a multigraph, and the author also calls them general graphs

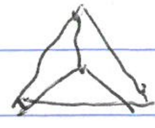
A graph is called complete if each pair of distinct vertices has an edge. Each vertex is incident to $n-1$ edges and there exist $\frac{n(n-1)}{2}$

edges since n even vertices have $n-1$ edges adjacent. But we have double counted, so the ~~answer~~ number is $\frac{n(n-1)}{2}$. Complete graphs are denoted

by K_n.

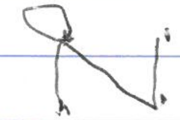
 is a K_4

Typically, in drawing graphs, edges intersect at points that are not vertices. K_5 has the property. A graph is called planar if the only places that edges intersect is at vertices. K_4 is planar and it seen as

 ~~or~~ as

In the definition of multigraph, (x,x) can appear. It is called a loop. The number of edges incident with a vertex is called the degree of the vertex. Anytime (x,x) appears it

counts twice in the count of the degree
Given any G, list the degrees in non-increasing order. This list is called the degree sequence of the graph.

Ex  Has degree sequence $(4, 2, 1, 1)$

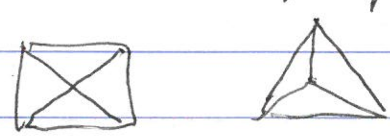
Ex K_n has degree sequence $(n-1, n-1, \dots, n-1)$ (n of these $n-1$'s)
Denote the degree sequence by
$$d_1, d_2, \dots d_n$$

Thm Let G be a general graph (including a graph) Then
1. $d_1 + \dots + d_n$ is an even number
2. the number of vertices of odd degree is even

Proof. 2 follows immediately from 1.
For 1, each edge is incident to 2 vertices (in the case of a loop, the one vertex counts twice)
So each edge counts twice in
$$d_1 + \dots + d_n$$
Hence this term is even.

Ex At a party, the number of guests that shake an odd number of hands is even. Consider the os the hands being vertices and and shaking as an edge (pair). Then the number of hands shaken is even and the number of hands of odd degree is even.

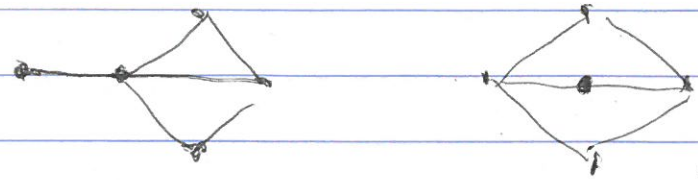
Are the graphs shown by

★ 

really different?

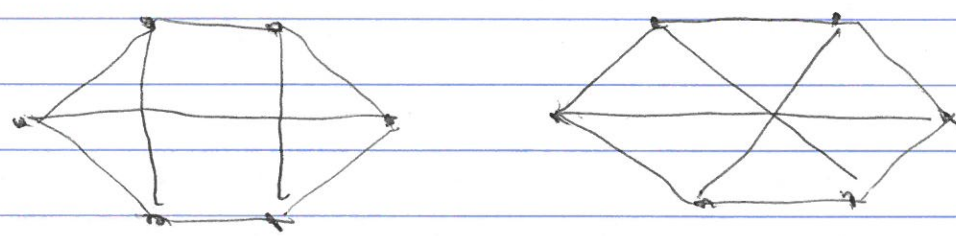
Def Graphs are isomorphic if there is a 1-1 correspondence Θ between their vertices such that $(x,y) \to (\Theta x, \Theta y)$. In other words, a bijection on the vertices that preserve the adjacent edges. One can see the graphs in ★ are isomorphic, in fact, any bijection between the vertices will preserve the edges.

If G and G' are isomorphic, clearly they have the same number of vertices and edges

Ex 

They have 5 vertices and 6 edges. They can not be isomorphic since the first graph has a vertex of degree 1 but the second one does not

It is evident that an isomorphism take vertices of the same degree to each other, hence isomorphic graphs have the same degree sequence.
The following example shows that non-isomorphic graphs can have the same degree sequence



Observation shows both graphs have degree sequence
3, 3, 3, 3, 3, 3

However they can not be isomorphic. In the first graph, there is a triangle on the left side (also on the right) The second graph does not have any triangles. There can be no isomorphism for there is no place for the triangle to go

Def. Let $G=(V,E)$ be a general graph
A sequence of edges
$(x_0,x_1),(x_1,x_2),\dots,(x_{m-1},x_m)$ is

1. called a walk of length m
Notation $x_0 - x_1 - x_2 \quad - x_m$
2. A walk is closed if $x_0 = x_m$
otherwise it is open
3. A walk with distinct edges is called a trail
4. If it is a trail and has distinct vertices (except, perhaps, x_0 and x_m) it is called a path
5. A closed path is called a cycle

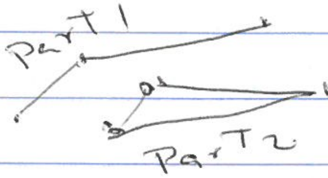
The length of a cycle is at least 3
A loop forms a cycle of length 2
An edge of multiplicity ≥ 2
determines a cycle $a-b-a$ of length 2

PROBLEMS p 449 2,3,4,5,6

Lesson 8

A general graph is called connected provided that, for each pair of vertices there is a path joining x and y. A graph can be partitioned into connected parts

Ex



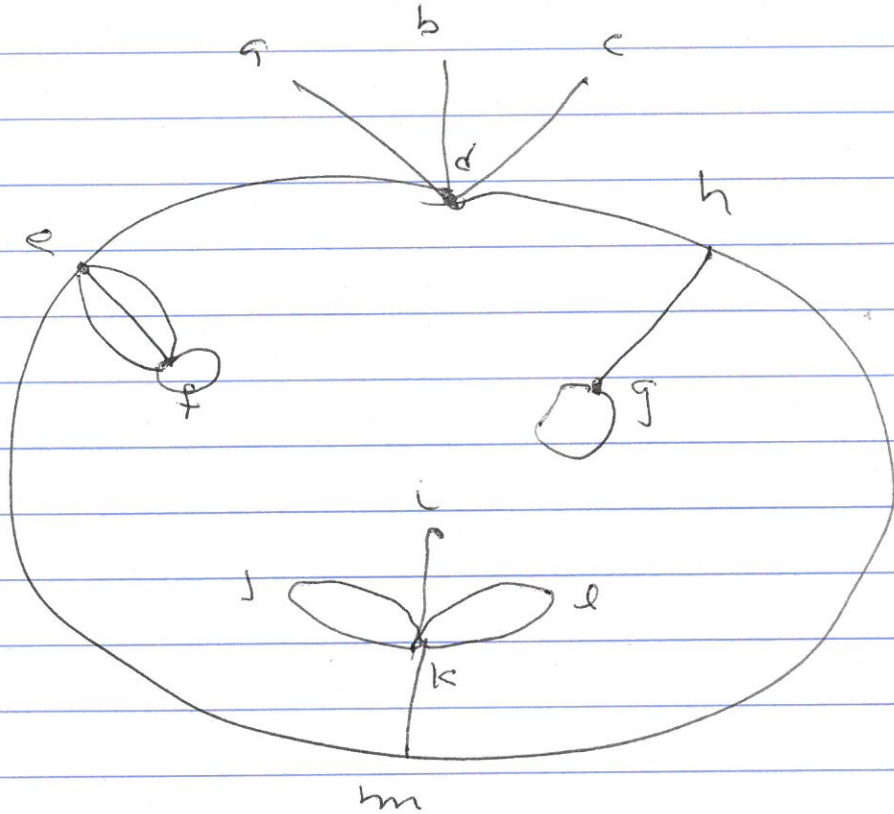
In a connected graph the shortest walk from x to y is called the distance from x to y (and y to x) and denoted by $d(x,y)$

Ex

$d(x,y) = 3$

A walk joining x to y of length $d(x,y)$ is a path

The following complex example is on page 398 and on some page after that. Brualdi calls it the GRAPHBUSTER



It is clearly a multigraph
It shows examples of

1. a — d — b — d — c — d — h — g — h — m — k

is a walk of length 11 but is not a trail

2. a — d — e — f — e — m — k — l — k — l

is a trail of length 9 joining a to l but it is not a path

3 a – d – e – m – k – i is a path of length 5 joining a to i

4 d – e – f – e – m – h – d
is a closed trail of length 6 but it is not a cycle

5. f – f, e – f e, d – e – m – h – d

are all cycles

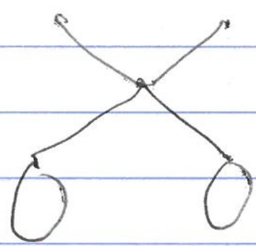
Continuing with this example in a few lines.

Let $G = (V, E)$ be a general graph
Let (U, F) be such that U is a subset of V and F is a submultiset of E such that the vertices of each edge in F are in E. Then (U, F) is also a general graph, called a subgraph (general subgraph) of E. If F consists of ALL edges in G that join vertices in U, the $G' = (U, F)$ is an INDUCED subgraph of G.
If U is all of V the G' is called spanning (spans G). An induced subgraph of G is obtained by selecting some of the vertices of G and all edges of G that join the vertices we chose. A spanning general subgraph

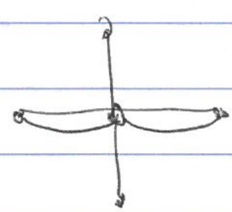
is obtained by taking all the vertices of G and some (or all) of the edges of G

The author phrases the example at the bottom of page 403 as Find the following in the GRAPHBUSTER and then gives an answer

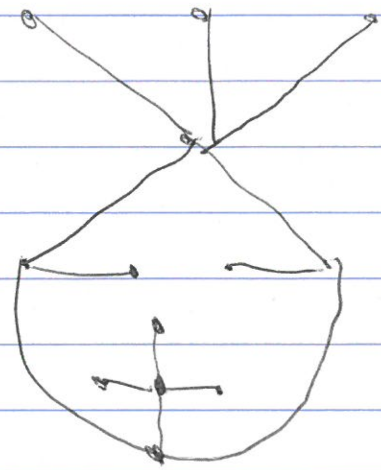
1. A general subgraph that is neither induced nor spanning



2. A general subgraph that is induced but not spanning



3. A general subgraph that is spanning but not induced



Make sure you see these to understand the concepts

The following Thm is obvious

Thm Let G and G' be general graphs. If they are isomorphic then

1. If G is a graph so is G' (remember this means no more than one edge between vertices) If one of G or G' has this property, so does the other

2 If G is connected, then so is G'. They have the same number of connected components in the partitions into connected components

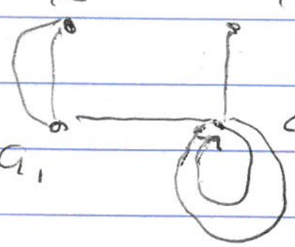
3 If G has a cycle of length K, then so does G' (Such a geometric condition has many relatives) like if one has

a subgraph that is a Δ, so does the other
Take 4
4 If G has an induced K_n then so do G'.

Adjacency Matrix

Let G be a graph with vertices $a_1 \ldots a_n$. Label the rows and columns with $a_1 \ldots a_n$. The entry in the (i,j) position is the number of edges between a_i and a_j and (i,i) is the number of loops at a_i. Clearly given a graph we can count the edges and get a_{ij} and given a_{ij} that is the number of edges from a_i to a_j. This matrix is a convenient way of keeping track of the edges in a graph

Ex



$$A = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

G is a graph is all $a_{ij} = 0$ or 1
G has a loop at a_i iff $a_{ii} \neq 0$

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G is connected if we can not rearrange the a. and get a block matrix (square matrices going down the diagonal and rest of matrix is 0)

Problems to be added