Lectures 5 and 6

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$$h_n = 8h_{n-1} - 16h_{n-2} \quad h_0 = -1 \quad h_1 = 0$$

$$h_n - 8h_{n-1} + 16h_{n-2} = 0$$

$$g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 +$$

$$-8xg(x) \quad -8xh_0 \quad -8x^2h_1 \quad -8x^3h_2 +$$

$$+16x^2 g(x) = \quad 16x^2 h_0 + 16x^3 h_1 +$$

$$(1 - 8x + 16)g(x) = h_0 + (h_1 - 8h_0)x + (h_3 - 8h_2 + 16h_0)x^2 +$$

$$= -1 + 8x + 0$$

$$g(x) = \frac{-1 + 8x}{(1-4x)^2} = \frac{A}{1-4x} + \frac{B}{(1-4x)^2}$$

$$-1 + 8x = A(1-4x) + B$$

$$-1 = A + B \qquad B = -3$$

$$8 = -4A \qquad A = -2$$

$$g(x) = \frac{-2}{1-4x} + \frac{-3}{(1-4x)^2} = -2\sum 4^n x^n + (-3)\sum \binom{n+1}{n} 4^n x^n$$

$$= \sum \left[(-2)4^n - 3\binom{n+1}{n}4^n\right] x^n$$

$$h_n = (-2)4^n - 3\binom{n+1}{n}4^n$$

Solve $h_n = 6h_{n-1} - 9h_{-2} + 2n \qquad h_0 = 1,\ h_1 = 0$

1. Homogeneous

$$h_n = 6h_{n-1} - 9h_{n-2}$$
$$x^n - 6x^{n-1} + 9x^{n-2} = 0$$
$$x^2 - 6x + 9 = 0$$
$$(x-3)(x-3) = 0$$
$$x = 3,\ 3$$
$$h_n = c_1 3^n + c_2 n 3^n$$

Particular $\quad h_n = An + B$

$$An + B = 6(A(n-1) + B) - 9(A(n-2) + B) + 2n$$
$$= 6An - 6A + 6B - 9An + 9A2 - 9B + 2n$$

n: $\quad A = 6A - 9A + 2$

$$B = -6A + 6B + 18A - 9B$$
$$4A = 2 \qquad A = 1/2$$
$$4B = (-6 + 18)A = 12A = 6$$
$$B = 2/3$$

$h_n = \frac{1}{2} n + 2/3$ is the particular soln

$$h_n = c_1 3^n + c_2 n 3^n + \frac{1}{2} n + 2/3$$
$$1 = c_1 + 2/3$$
$$0 = 3c_1 + 3c_2 + 1/2 + 2/3$$
$$c_1 = 1/3$$
$$0 = 1 + 3c_2 + \frac{7}{6} \qquad 3c_2 = -13/6$$
$$c_2 = -13/18$$

$$h_n = \frac{1}{3} 3^n + (-13/18) n 3^n + \frac{1}{2} n + 2/3$$

More Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

C_n satisfies a recurrence relation

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\,\frac{(2n)!}{n!\,n!}$$

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} = \frac{1}{n}\,\frac{(2n-2)!}{(n-1)!\,(n-1)!}$$

$$\rightarrow \quad \cancel{\frac{C_n}{C_{n-1}} = \frac{(2n)!}{n!\,n!}\quad\frac{(2n-2)!}{n}}$$

$$\frac{C_n}{C_{n-1}} = \frac{1}{n+1}\,\frac{(2n)!}{n!\,n!}\,\frac{n\,(n-1)!\,(n-1)!}{(2n-2)!}$$

$$= \frac{n}{n+1}\,\frac{2n\,(2n-1)}{n\cdot n} = \frac{4n-2}{n+1}$$

$$\rightarrow \quad C_n = \frac{4n-2}{n+1}\,C_{n-1} \qquad C_0 = 1$$

is a recurrence relation for C_n

Now define the Pseudo Catalan Numbers

$C_1^* \quad C_2^* \quad C_3^* \cdots \quad C_n^* \cdots$

Let $C_n^* = n!\, C_{n-1}$
$\rightarrow \quad C_1^* = 1$

$$C_n^* = n!\, C_{n-1} = n!\, \frac{(4n-6)}{n} C_{n-2}$$
$$= (4n-6)(n-1)!\, C_{n-2} = (4n-6)\, C_{n-1}^*$$

$\rightarrow \quad C_n^* = (4n-6)\, C_{n-1}^*, \qquad C_1^* = 1$
is a recurrence relation for C_n^*

$C_1^* = 1,\ C_2^* = 2,\ C_3^* = 12,\ C_4^* = 120\ \ C_5^* = 1680$
is found using the recurrence relation

When multiplying n elements, there are many ways to do the multiplication. As a simple example, let $n = 3$, a_1, a_2, a_3 are the elements. Then

$a_1(a_2a_3)$ $a_2(a_3a_1)$ $a_3(a_1a_2)$
$a_2(a_1a_3)$ $a_1(a_3a_2)$ $a_3(a_2a_1)$
$(a_1a_2)a_3$ $(a_2a_3)a_1$ $(a_3a_1)a_2$
$(a_2a_1)a_3$ $(a_1a_3)a_2$ $(a_3a_2)a_1$

Giving 12 ways. The order of the elements and where the parentheses are both play into the complexity of the problem. Now let's insert a_4 into one of them, $a_1(a_2a_3)$. We see

$a_4(a_1(a_2a_3))$ $(a_4a_1)(a_2a_3)$
$(a_1a_4)(a_2a_3)$ $a_1(a_4(a_2a_3))$
$a_1((a_4a_2)a_3)$ $a_1((a_2a_4)a_3))$
$a_1(a_2(a_4a_3))$ $a_1(a_2(a_3a_4))$
$a_1((a_2a_3)a_4)$ $(a_1(a_2a_3))a_4$

Expanding one element of 3 terms to one of 4 terms gives 10 new terms. Each of the three original elements expands in the same number, 10, ways. Let h_n be the number of such schemes. Then $h_3 = 12$, $h_4 = 10h_3 = 120$

We claim $h_n = (4n-6)h_{n-1}$. Thus $h_5 = 14 \cdot 120$

Notice that $h_1 = 1$, $h_2 = 2$, $h_3 = 12$

There are, in the listing of n elements, $n-1$ multiplications and $n-1$ parentheses if we put parentheses around the outside.
To find h_n from h_{n-1}:
For any term in the $n-1$ case. Two possibilities arise on the insertion

1. In a term of $n-1$ elements, we insert a_n on either side of one of the $n-2$ multiplications (there are $n-2$ since this is $n-1$ elements)
   Each term in the $n-1$ case gives $2 \cdot 2 \cdot (n-2) = 4(n-2)$ ways

2. a_n is placed on the outside (left or right) giving 2 terms. Step 1 takes care of the rest.

So each of the h_{n-1} terms gives

$$4(n-2)+2=4n-6$$

terms using the method. Hence

$$h_n = (4n-6)h_{n-1} \qquad h_1 = 1$$

$$\rightarrow \quad h_n = C_n^* = (n-1)! \binom{2n-2}{n-1}$$

since h_n satisfies the same recurrence relation as C_n^* and the same initial condition

Suppose we count only those terms in which the numbers are listed in natural order, (but different places for the parentheses) We only consider ~~the~~ one of the $n!$ ways of ordering the n elements (letting g_n be the number of these terms)

$$g_n = \frac{h_n}{n!} = \frac{C_n^*}{n!} = \frac{1}{n!}(n-1)!\binom{2n-2}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

$= C_{n-1}$. So the Catalan numbers count the number of ways to put the parentheses (the elements are always in order $a_1 \dots a_n$)

i.e. $((a_1 a_2) a_3) a_4 \quad (a_1(a_2 a_3)) a_4 \quad a_1(a_2(a_3 a_4))$ would be examples of this new kind of count.

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