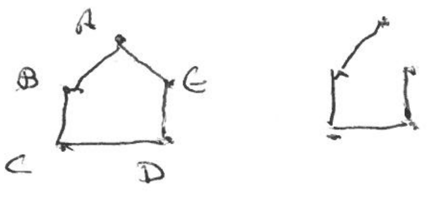
Connected graphs

A graph is connected if for every pair of distinct verticies there is a walk from one to the other



Are both connected

Any graph can be partitioned into subgraphs $G_1, .. G_K$, each of which is connected and there is no edge from any vertex in G_i to any vertex in G_j, $i \neq j$. The G_i are found by taking any vertex i and G_i will be all vertices with walks to i. Then take $j \notin G_i$ and repeat the process. Continue until all vertices are used.

1. What is the maximum number of edges in a graph of order n
2. Suppose G is partitioned into connected components of order s and t. What is the max number of edges in G?

Eulerian Trails

Recall: A sequence of edges

$$\{x_0, x_1\} \; \{x_1, x_2\} \cdots \{x_{n-1}, x_n\}$$

is called a walk of length n

We will write this as

$$x_0 - x_1 - x_2 - \cdots - x_n$$

It is closed if $x_0 = x_n$, otherwise it is open.

If a walk has no repeated edges, then it is called a trail

If it also has distinct verticies, except perhaps $x_0 = x_n$, then it is called a path

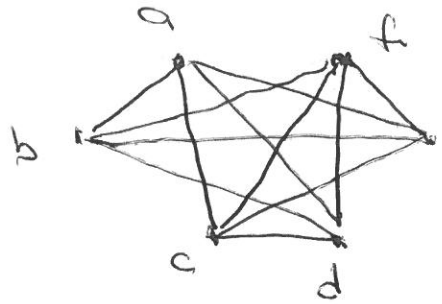
The definitions apply to multigraphs (also called general graphs) as well

In a general graph, a trail can have edges repeated up to the number of times they appear in the graph

A trail is called Eulerian if it contains every edge in the graph

(2)

Ex



Closed Trails

a - c - d - a

a - c - d - a - e - b - a

a - c - d - f - e - c - f - b - a - d - b - e - a

The last one is Eulerian

Ex

Multigraph

Closed Trail a - b - a

Open Trail a - b - a - b - a

There is no closed Eulerian Trail as not every edge can be used once and end up where we started

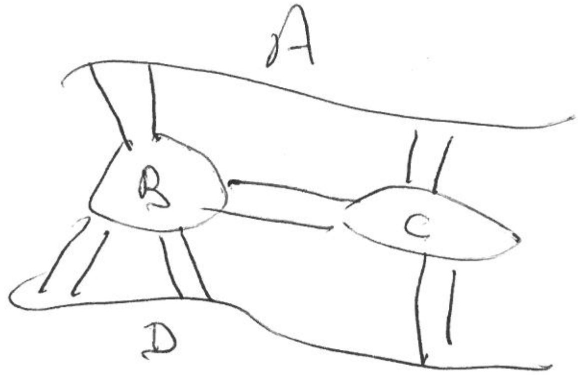
The last remark extends to necessary conditions for a graph to have a closed Eulerian trail, namely, each vertex has to have an even number of edges incident to it. The veertex

(3)

where we start gets one edge in the start. As we pass through vertices, the number of edges incident to it increases by 2, one coming and one going. The last vertex gets one more edge. The only way it is a closed trail is for $x_0 = x_n$, the only time the number of edges incident to x_0 (and x_n) is even. Hence the result. Recall, the degree of a vertex is the number of edges incident to it

This leads us to the problem that started all these ideas. It is due to Euler, hence the name Eulerian

(4) The KONIGSBERG bridge problem:
The city of Konigsberg has the map



where the double lines stand for a bridge and each bridge connects a land mass

The problem is: Can we start at a land mass (A, B, C or D) travel across each bridge exactly once and end up at the beginning land mass?

Considering this as a graph with verticies A, B, C and D and edges the bridges, we are asking is There a Clossed Eulerian Trail
The answer is no since the degree of B is odd (as are the

degrees of C and D

The necessary condition we found is also sufficient. for connected graphs (Graphs for which every pair of distinct vertices has a walk from one to the other) Hence the degree of each vertex is at least one in a connected graph. First a Lemma

Lemma Let $G = (V, E)$ be general graph and assume the degree of each vertex is even Then each edge belongs to a closed trail

Proof Pick any vertex and edge incident to it, say edge $\alpha_1 = (x_0 - x_1)$ x_1 has an even number of edges so there is an edge leaving x_1 for, say, x_2

$x_0 - x_1 - x_2$

Continue this. The final vertex, which

has even degree, has an odd number of edges incident to it in our trail and so there is another edge leaving it, unless it is the beginning vertex. So the process continues until we return to where we started with a closed trail in hand.

Note that the proof is an algorithm that allows us to find the objective, a closed trail. Always a valuable kind of proof.

Looking at the first example in this chapter, we see that the algorithm is used to construct the closed trails. Note also that the degree of each vertex is even.

There is a converse to the first theorem

⑦

Theorem. Let G be a connected general graph. Then G has a closed Eulerian Trail if and only if the degree of each vertex is even

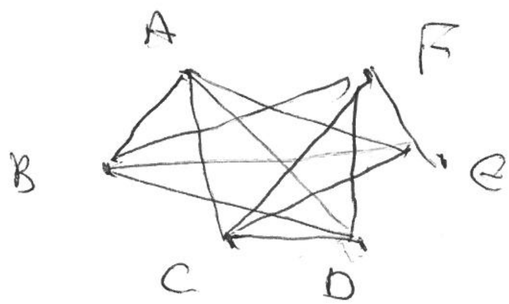
Proof. The first Theorem has shown that the existence of a closed Eulerian trail forces the degrees of the verticies to be even

Conversely, suppose the each vertex has even degree. Pick a vertex and apply the lemma to get a closed Trail (In our first example, that would be the first case) If not all edges have been used, remove the used edges. The verticies in the resulting new graph all have even degree The graph is connected so there is an edge incident to one on the closed trail. Thus, if z_1 is that vertex and some vertex z_2 can be connected to it by a non used edge since they have been removed. All vertices on the ~~su~~ left over subgraph have even degree, so there is a closed trail starting at z_1.

Insert this new trail into the old trail at z_1 and continue the process until all edges have been used.

This is finally a closed Eulerian trail

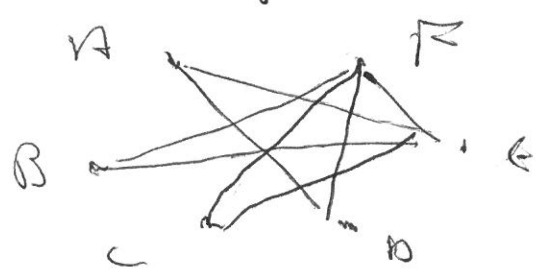
· Return to our example



Pick edge AC There is a closed trail

A − C − D − B A

Remove edges



Note all vertices still have even degree

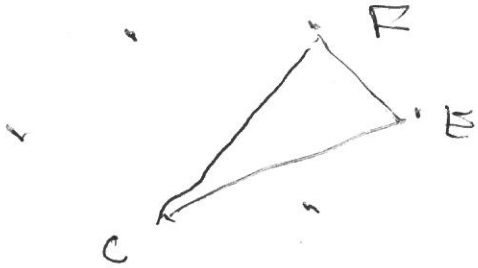
Pick a vertex on the trail with an edge not used Pick d-f

There is a closed trail a-f-b-e-a-d

Insert it into the original

a-c-d-f-b-e-a-d-b-a

Remove edges



Use trail F-C-E-F

Insert at F

a-c-d-f-c-e-f-b-e-a-d-b-a

All ~~trails~~ edges have been used once.
This is a closed Eulerian Trail

Def An open Eulerian Trail is one that uses all edges exactly once but is not closed (Evidently it lacks one last edge from being

(10) a closed Eulerian Trail

Theorem. Let G be a connected graph G has an open Eulerian Trail if and only if there are exactly two verticies of odd degree

Proof. An open Eulerian Trail have each vertex, except first and last, have even degree (as always on any trail). The first and last have odd degree as explained in the other proofs.

Conversely, if the condition holds Suppose u and v are the two vertices of odd degree. Form a new graph with one new edge from u to v. Now all vertices have even degree. There is a closed Eulerian Trail by the last theorem. On this trail remove the edge u-v. This is an open Eulerian Trail with end points u and v

Problems

PAGE 450:

2 3 4 5 11 13 14 15 20 21