# Chromatic Number

Def. A graph $G=(V,E)$ is bipartite if it's vertex set can be partitioned into sets $X$ and $Y$ such that there are no edges between elements in $X$ and no edges between elements in $Y$. A graph with no edges is called a null graph

Ex $G=(V,E)$ $V=\{1,2,\cdots,10\}$ For $a,b\in V$ $a-b\in E$ if and only if $a-b$ is odd
$X=\{2,4,6,8,10\}$ $Y=\{1,3,5,7,9\}$

Theorem $G$ is bipartite if and only if each of it's cycles has even length. We are assuming that $G$ is connected.

Proof If $G$ is bipartite with sets $X$ and $Y$ as in the definition. Then any cycle must alternate between vertices in $X$ and vertices in $Y$ Since it must end at the vertex that it started at, the cycle must be even

Suppose each cycle has even length.
Pick $x \in V$. Let $X$ = all $v \in G$ whose shortest distance to $x$ is even and $Y$ = all $v \in G$ whose shortest distance to $x$ is odd. $X$ and $Y$ partition $V$.
We want to show that no two vertices in $X$ ($Y$) are adjacent.
For a contradiction, suppose $a, b \in X$ and $a - b$ is an edge. Let

$\alpha: x - \cdots - a \qquad \beta: x \cdots b$

be walks of the shortest distance ~~to~~ from $x$ to $a$ and $x$ to $b$.
There is a last common element $z$ in these walks (it might be $x$)

$\alpha: x \cdots z \cdots a \qquad \beta: x \cdots z \cdots b$

Consider $\alpha_1: x \cdots z \qquad \alpha_2: t \cdots a$

$\beta_1: x \cdots z \qquad \beta_2: t \cdots b$

Claim $\alpha_1, \beta_1$ have the same length
If not suppose $\alpha_1$ is shorter. Then $\alpha_1, \beta_2$ is shorter from $x$ to $b$ than $\beta_1, \beta_2$ is, a contradiction. Since $\alpha$ and $\beta$ are of even length ($a, b$ are in $X$), $\alpha_2$ and $\beta_2$ are both even or both odd length. Then

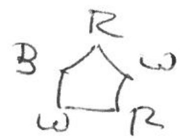
$$\underbrace{z \cdots a}_{\alpha_2} - \underbrace{b \cdots t}_{\beta_2} \text{ is of odd length}$$

and is a cycle. A contradiction. Hence there is ~~no od~~ $X$ is a null graph ($Y$ works the same)

# CHROMATIC NUMBERS

Def Let $G$ be a graph and $S$ be a set with $k$ elements (called colors but they need not be colors). A coloring of $G$ is an assignment of elements from $S$ to each vertex in $V$ such that no adjacent vertices get the same color. It is also called a $k$-coloring of $G$.

Ex $S = \{R, W, B\}$ $G =$ 

Coloring 


What is the smallest $k$ such that there is a coloring of $G$? This number is called the chromatic number of $G$ and denoted by $\chi(G)$.

Ex If $G$ is the null graph, $\chi(G) = 1$
If $G = K_n$, $\chi(G) = n$ since every vertex is ~~connec~~ adjacent to every other vertex 

(3 colors won't do)

Remarks. $|V| = n$.

1. If $|V| = n$, then $1 \leq \chi(G) \leq n$
2. $\chi(G) = 1$ if and only if $G$ is the null graph
3. $\chi(G) = n$ if and only if $G = K_n$

4)

4 If $G = C_n$, an even cycle, then $\chi(G) = 2$

5 If $G = C_n$, an odd cycle, then $\chi(G) = 3$

6 If $G$ = tree, then $\chi(G) = 2$

Proof 1. This is clear

2. If G is the null graph, then there are no edges, so each vertex can get the same color. If G is not the null graph, there is an edge and 2 colors are necessary

3. If $G = K_n$, we have noted $\chi(K_n) = n$. If $G \neq K_n$, there are 2 vertices, ~~with~~ not adjacent, so they can get the same color

4. If $G = C_n$, $n$ even, alternating 2 colors will work

5 If $G = C_n$, $n$ odd, alternating 2 colors fails at the last vertex as there is ~~one~~ different colors on the vertices adjacent to it. A third color will work

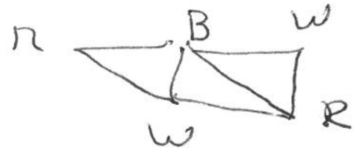
6. Pick a vertex and color it. Then as you move along any branch, alternate the 2 colors. This works and $\chi(G) = 2$

5/ If $H$ is a subgraph of $G$, then any coloring of $G$ is also a coloring of $H$, hence $\chi(H) \le \chi(G)$

Ex  A subgraph is seen to be $K_3$

$\chi(G) \ge \chi(K_3) = 3$

We can then complete the coloring




$\chi(G) = 3$

As often in math, we try to break the problem into simpler cases. Suppose $G = (V, E)$ is $k$-colored with colors $1, 2, \dots, k$. Let $V_i = \{v \in V;\ v \text{ is colored with color } i\}$. Then $V_i$ is a null graph. Each vertex in $V_i$ can be colored by the same color. Hence $\chi(G) \le k$. (We might do better with a different coloring) The set $\{V_i\}$ is called a color partition of $G$.

What is the color partition in the example at the top of the page?

Theorem. Let $G = (V, E)$ $|G| = n$ and $q$ be size of $V_1$, the largest subgraph of $G$ that is a null graph. Let $\chi(G) = k$ and $V_1, .. V_k$ be the color partition of $G$.

Then $n = |V| = |V_1| + \ + |V_k| \le |V_1| + \ + |V_k| = kq$

So $\chi(G) = k \ge \frac{n}{q}$

Ex In our example above, $n = 5$, $q = 2 \to \chi(G) \ge 5/2 \to$ $\chi(G) \ge 3$

6)

Thm. Suppose G has at least one edge. Then $\chi(G) = 2$ if and only if G is bipartite.

Proof. If G has a bipartite partition X, Y, then we can color all vertices in X with one color and all vertices in Y with a second color, $\chi(G) = 2$.

If $\chi(G) = 2$, put all elements of the first color in X and all elements of the second color in Y. There are no edges between elements in X (or Y) so this is a bipartite partition.

Corollary. Suppose G is not the null graph. Then G is $\chi(G) = 2$ iff and only if every cycle is of even length.

Proof. Both these conditions are the same as G having a bipartite partion (from the above result and the first result today)

7) The following coloring algorithm describes a coloring algorithm that gives an upper bound for $\chi(G)$. It is tied to the degrees of the vertices, in particular the maximum degree, $\Delta$

Let $\{x_1, \ldots x_n\} = V$

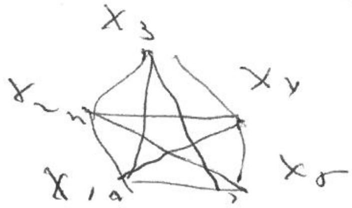
Assign 1 to $x_1$

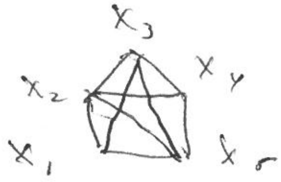
For $x_2$ assign the smallest number not used in $\{x_1\}$. Continue

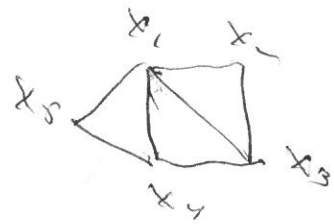
For $x_j$ assign the smallest number not used in $\{x_1, \ldots x_{j-1}\}$

As is clear, when we get to $x_j$, there are $j-1$ vertices before it. At most $j-1$ number have been assigned so $x_j$ can at worst get $j$. But the condition on $\Delta$ says there are at most $\Delta$ edges to elements before $x_j$ (for any $j$), so at most $\Delta$ ~~num~~ colors have been assigned. The worst $x_j$ can get is $\Delta+1$. For every $j$ since $\Delta$ is the most number of edges adjacent to any vertex

Examples




| Vertex | Color |
|---|---|
| $x_1$ | 1 |
| $x_2$ | 2 |
| $x_3$ | 3 |
| $x_4$ | 4 |
| $x_5$ | 5 |

$\Delta = 4$

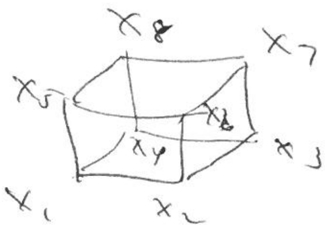
| Vertex | Color |
|---|---|
| $x_1$ | 1 |
| $x_2$ | 2 |
| $x_3$ | 3 |
| $x_4$ | 1 |
| $x_5$ | 4 |

$\Delta = 4$

| Vertex | Color |
|---|---|
| $x_1$ | 1 |
| $x_2$ | 2 |
| $x_3$ | 3 |
| $x_4$ | 2 |
| $x_5$ | 3 |

$\Delta = 4$

Try this for $G = (V, E)$ $V$ = vertices of a cube, $E$ = edges




What is $\Delta$? What is $\chi(G)$!

Do you see one of our Theorems?

Notice: This algorithm gives a bound on $\chi(G)$. It might be far smaller than $\Delta + 1$. In fact, the only connected graphs which hit this bound are all $K_n$ and $C_n$, $n$ odd