Chromatic Polynomial

Let $G=(V,E)$, $|G|=n$. Given k, how many k colorings exist. Call this $P_G(k)$.

If $k < \chi(G)$, then none exist, so $P_G(k)=0$.

Ex. Let G be a tree. Pick any vertex, v, it can be colored by any of the k colors. Each vertex adjacent to v can be colored by $k-1$ colors each. Following this pattern each of the next adjacent vertices can receive $k-1$ colors. and so on. Hence $P_G(k)=k(k-1)^{n-1}$

Ex Let G be N_n the null graph with n vertices. Clearly $P_G(N_N)=k^n$

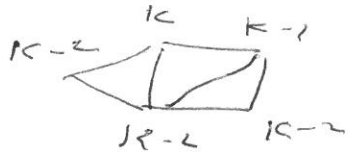
Ex $G=K_n$. A first vertex can have k colors, a second has $k-1$ choices. Since each new vertex is adjacent to all the previous ones, it can not get one of these colors. If $v_1, \ldots v_n$ are the vertices:

TABLE	Number of colors
v_1	k
v_2	$k-1$
v_3	$k-2$
v_4	$k-3$

v_n gets $k-n+1$ choices

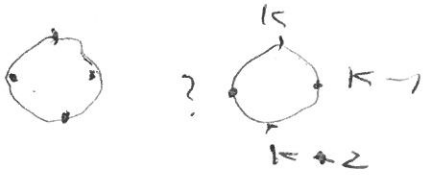
Hence $P_G(k) = k(k-1)\cdots(k-n+1) =$

the number of permutations of k elements

Ex 

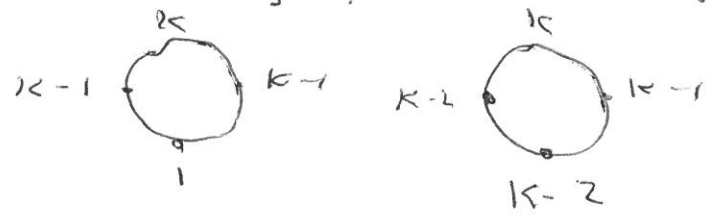
$P_G(k) = k(k-1)(k-2)^3$

When cycles exist (ie not Trees) these assignments can get Tricky

Ex 

What goes in? It depends

If the object at top and bottom are different? $= k-2$ If not? $= k-1$



$P_G(k) = k(k-1)^2 + k(k-1)(k-2)^2$

As you can guess, if this simple example requires cases, things can get complicated

By the way, in our examples, $P_G(k)$ is a polynomial. This is always the case. It is called The chromatic polynomial for G and its values for each k gives the number of colorings

There is an algorithm to compute $P_G(k)$. Each step has 2 parts, so it can be quite lengthy. The idea is to reduce the graph to a collection of null graphs, with varying number of vertices which number gives a monomial in $P_G(k)$. After all is reduced there might be, say, 7 null graphs with 4 vertices. This contributes k^4. If there are 5 of them, we get $5k^4$. It is actually $\pm 5k^4$ and a miracle in this algorithm is that some of the k^4's ~~hav~~ don't have different signs. They are all either + or - $P_G(k)$ shows many properties of the graph as we will see.

Algorithm for $P_G(k)$

We do this algorithm until there are no edges left!

1. Pick x and y such $x-y \in E$ □
2. Remove $x-y$ to get a new graph G_1 □
3. G_1 has 2 possibilities, either ~~The~~ x and y are colored different or colored the same (In G_1 not G)

4) 4. Let $C(k)$ = graphs in G_1, x and y different
$C(k)$ is in 1-1 correspondence with
graphs coloring in G

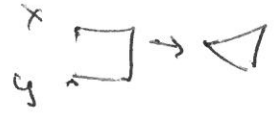
5 Let $D(k)$ = graphs in G_1, x and y colored
the same

Clearg $P_{G_1}(k) = C(k) + D(k)$

$P_{G_1}(k) = P_G(k) + D(k)$

$\rightarrow P_G(k) = P_{G_1}(k) - D(k)$

6 In The graphs in $D(k)$, fuse x and y
with all edges aligning



Notice the number of edges in
both $P_{G_1}(k)$ and $D(k)$ are less then
those in $P_G(k)$. This means if
we repeat this enough times
(the number of edges to start
with would be an upper bound)
we get null graphs, just what
we want. So $P_G(k)$ = combination of
monomials obtained from these null
graphs. Also note that the number
of vertices does not drop in $P_{G_1}(k)$
but drops by 1 in $D(k)$
Suppose we stut with 7 vertices and
in one of these sequences end with

a null graph with 4 vertices
It has taken 7-4 sign changes
to go from 7 to 4 so the
sign on the monomial k^4 will
be negative when we plug it
back into this huge expanded
expression. And it happens each
time we go from 7 vertices to 4
All In general, this always happen
The monomial x^s in the final expression
has coefficient = the number of
null graphs with 4 vertices and
its sign is the difference t-s when
we have t vertices in G.

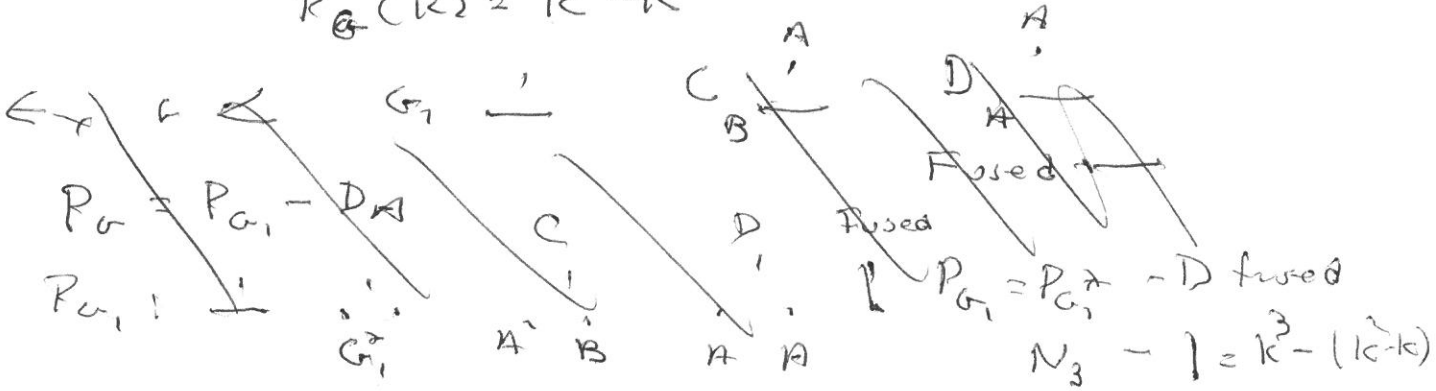
Ex G — G_1 : C : D :
A B A A
Fuse

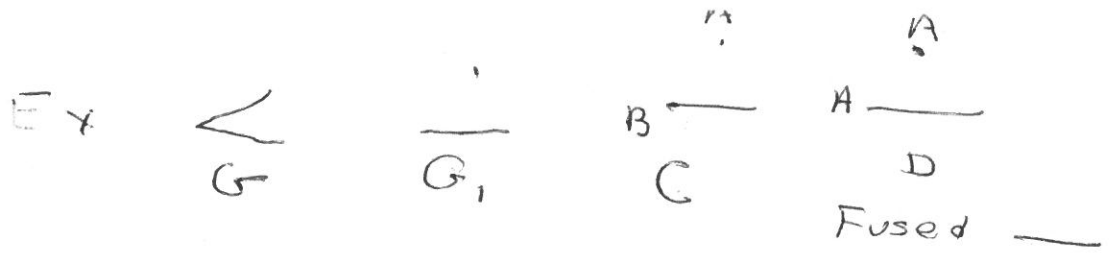
$G_1 = C + D$

$G_1 = G - D$

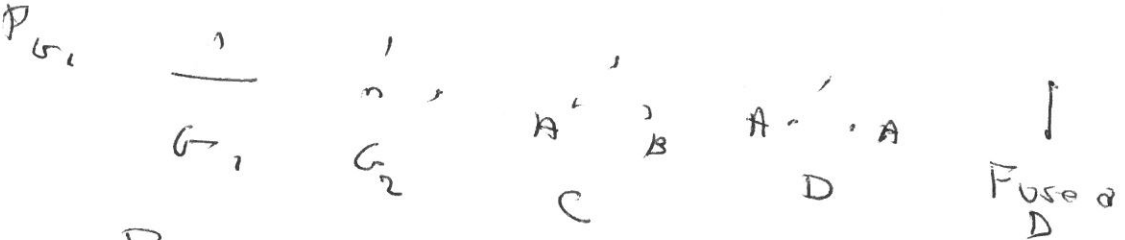
$P_G(k) = P_{G_1}(k) - D(k)$

we are already There
G_1 null graph 2 vertices k^2
D fused null graph 1 vertex
$P_G(k) = k^2 - k$



Ex 

$P_G = P_{G_1} - D_{fused}$

P_{G_1} 

$P_{G_1} = P_{G_2} - D_{FUSED}$

G_2 is a null graph, 3 vertices k^3

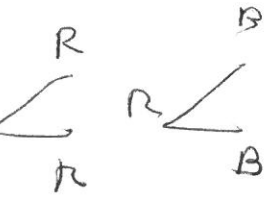
D_{FUSED} Was computed in last example k^2-k

$P_{G_1} = k^3-(k^2-k)$

First Defused → Again get k^2-k

$P_G = \underbrace{k^3-(k^2-k)}_{P_{G_1}} - \underbrace{(k^2-k)}_{D_{fused}}$

$= k^3-2k^2+2k$

$P_G(2) = 8-8+2=2$ colorings 

It is clear this gets complicated fast. A computer algebra package would have it programmed. What the algorithm does for us is shows we get a polynomial after some steps

First, the degree $P_G(k)$ = number of vertices

Second, the constant in the polynomial is 0 since we have added null graphs to set it and they all have a k in them

Third In a connected graph, one of the null graphs has one vertex so k appears as a first degree term. The coefficient is the number of null graphs with one vertex and the sign is obtained from $(-1)^{|V|-1}$ if even, $+1$
If odd, -1

If G has just two connected components G_1 and G_2 then

$$P_G(k) = P_{G_1}(k)\, P_{G_2}(k)$$

This is argued by doing the reduction first on G_1 and then on G_2
Requires some details but not much
In this case, $P_G(k)$ will have a k^2 term but no k term. Check what

happens when you multiply 2 polynomials with k terms but no constant. More generally, given a connected components, we get a k^2 term but no smaller

The polynomial tells us the number of connected components

Also, ↓ spectacularly

The coefficient of k^{n-1} = $-m$ where m is the number of edges in the graph

The algorithm and induction show this (Induction on number of edges)

$$P_G(k) = P_{G_1}(k) + P_{\text{fused}}$$

If G has m edges

G_1 has $m-1$ edges

So $P_{G_1}(k) = k^n - (m-1)k^{n-1} + \dots$

$P_{\text{fused}} = k^{n-1} + \dots$ (one vertex gone)

$P_{G_1}(k) - P_{\text{fused}} = k^n - m k^{n-1} + \dots$

9)

Notice also that in the examples if you multiply them out, the signs of the monomials alternate. This is explained by alternating signs as we eliminate vertices in the algorithm and is the same argument as we have already used

This is just fantastic stuff!

We will compute some chromatic polynomials by using the first step of the algorith to reduce the problem to known graphs:

Ex 1 

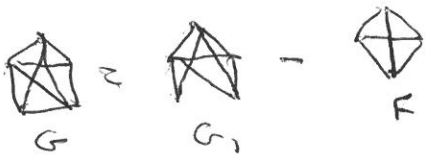
$$P_G = P_{G_1} - P_F = k(k-1)^3 - k(k-1)(k-2)$$
$$= k^4 - 4k^3 + 6k^2 - 3k$$

Ex 2 

$$P_G = P_{G_1} - P_F = k(k-1)^4 - (k^4 - 4k^3 + 6k^2 - 3k)$$

Ex 3 

$$P_G = P_{G_1} - P_F = (k^4 - 4k^3 + 6k^2 - 3k) - k(k-1)^2$$

Ex 4 

G_1 is K_4 with an extra vertex and 2 edges

$P_{G_1} = k(k-1)(k-2)(k-3)(k-2)$

$P_F = k(k-1)(k-2)(k-3)$ (it is a K_4)

$P_G = P_{G_1} - P_F = k(k-1)(k-2)(k-3)[k-2-1]$

Ex 5 If G is a n-cycle, then

$$P_G(k) = (k-1)^n + (-1)^n(k-1)$$

Induction on n

$n=3$  $= k(k-1)^2 - k(k-1)$

$= (k-1)^2 - (k-1)$

Assume for $G = C_{n-1}$ $P_G = (k-1)^{n-1} + (-1)^{n-1}(k-1)$

then

$$P_{C_n} = k(k-1)^{n-1} - [(k-1)^{n-1} + (-1)^{n-1}(k-1)]$$
$$= (k-1)^n + (-1)^n(k-1).$$

Problems Page 498-500

4, 5, 6, 9, 10, 11, 16

Exercises PAGE 498-500

1. Chromatic Numbers: 4, 5, 6, 10

2. Chromatic Poly 9, 11, 14, 16
   In these poly problems use either the properties of the chromatic poly or the First step in the algorithm Thm. (and maybe induction)