Trees

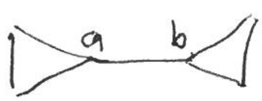
Let G be a connected graph, $|G| = n$. Consider the following three properties:

I The number of edges is $n-1$

II There are no cycles in G

III Every edge is a bridge

(Recall that a bridge in a connected graph is an edge which, if removed, leaves the graph not conneted



Edge a-b is a bridge.

Theorem. The three conditions are equivalent

Proof I → II.

Suppose $\gamma: x_0 - x_1 - \quad - x_k = x_0$ is a cycle. All the x_i are by definition distinct. There are k edges and k verticies. Each $y \in G$, y not on γ is connected to γ using at least one edge. There are $n-k$ verticies not on γ, so there are $n-k$ edges used. at least Counting the edges in γ, there are at least n edges, a contradiction

II → III

Suppose $\alpha: u - v$ is not a bridge. Removal of α leaves the graph, G^α connected, hence there is a path $\gamma: v - x_1 - \quad - x_k - u$ in G^α

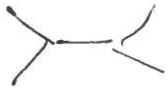
Then $v - x_1 \quad - x_k - u \to v$ is a cycle in G, a contradiction

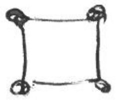
$III \to I$

Proceed by induction on n. If $n=2$, both III and I hold clearly. Assume for all cases less than order n. Let u-v be an edge, hence a bridge. The subgraphs on each side of the bridge, call G_1, G_2, $|G_1|=k$ $|G_2|=n-k$. Each edge is a bridge in G_1 and in G_2. Hence the number of edges in G_1 is $k-1$, in G_2 is $n-k-1$. Then the total number of ~~bridges~~ edges in G is $(k-1)+(n-k-1)+1=n-1$ where the last 1 is the edge that was removed.

Hence the Theorem is true

Def A connected graph with any, hence all, of these properties is called a TREE

 is a Tree $|G|=6$ and the number of edges is 5

 is not a tree. There is a cycle, no edge is a bridge and $|G|=4$ = number of edges

 is a Tree

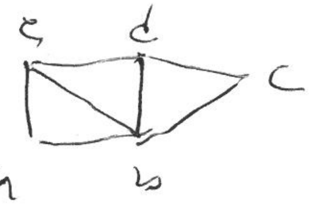
There are 2 trees when $n=4$

Let G be a connected graph and u_1 be a vertex. Let G_1 be the subgraph with vertex u_1. Since G is connected, There exists a u_2 which has an edge with a vertex in G_1. Let G_2 be the graph G_1 with the new vertex and edge. G_2 has 2 verticies and 1 edge. Repeat

This to get G_3 with 3 verticies and 2 edges, a tree. We ca continue up to G_n, n vertices and n-1 edges. G_n and G have the some vertices but G_n is a tree. G_n is called a spanning tree for G and every connected graph has them as we have just seen. This also gives a technique for constructing them. G could have many spanning trees

Ex



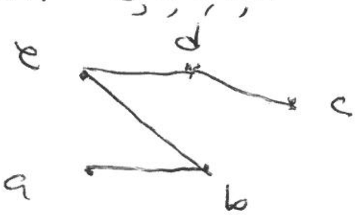
Pick e

Then e,b   e-b

Then e,b,a   e-b b-a

Then e,b,a,d   e-b b-a e-d

Then e,b,a,d,c   e-b b-a e-d d-c



Spanning Tree

We construct trees by starting with a vertex. Pick another vertex and connected the two with an edge continue this process and get a tree of any order desired

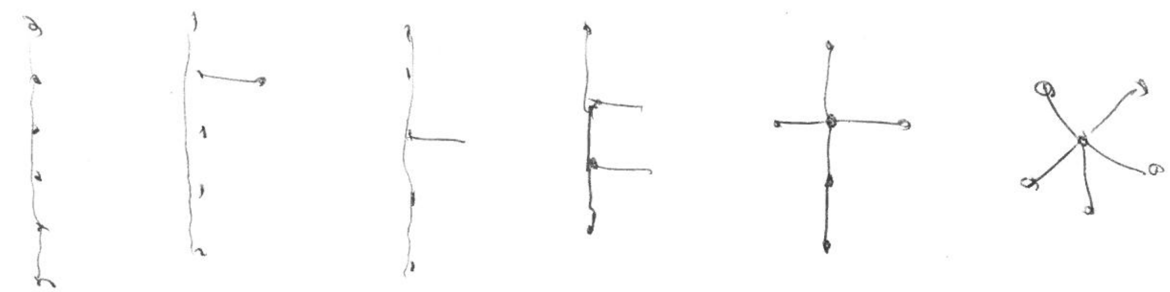
Ex  $n=5$

Sereal of The vertices have degree 1 Such a vertex is called a pendent vertex. It's edge is called a pendent edge

Trees of the type  have 2 pendent ~~edges~~ verticies. Trees of the type  have $n-1$ pendent ~~edges~~ verticies. These clearly are bounds:

A Tree of order n have between 2 and $n-1$ pendent verticies

In constructing all trees of a certain order, it is useful to track the degree sequences to see if the trees are not isomorphic For instance, the trees of order 6



~~1 2 2~~
2,2,2,2,1,1   3,2,2,1,1,1   3,2,2,1,1,1   3,3,1,1,1,1   4,2,1,1,1   5,1,1,1,1

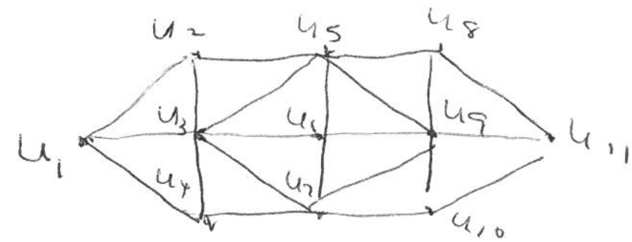
The degree sequences tell that all are not isomorphic except, perhaps, the 2nd and 3rd which would need to be resolved another way. It is clear there is no bijection between verticies that keep the edges aligned.

Various methods exist to construct spanning trees that have special properties. We will content ourselves by looking at one of them to get the general idea. We look at the so called breadth-first spanning trees. They are rooted (start at) a choosen vertex u in the vertex set U. The property they have is the distance between x in the graph and u is the same distance (shortest) in G and in the spanning tree, T. Two numbers are attached to each $x \in G$: The breedth first number $bf(x)$ which stand for the order in which x was added in the algorithm. The second number

This continues until no more y's exist. ~~If~~ In the end $U = V$ if and only if G was connected.

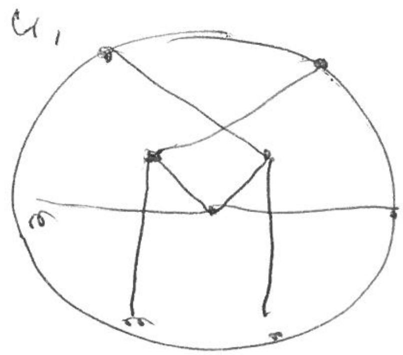
The proof, in the text, is straightforward.

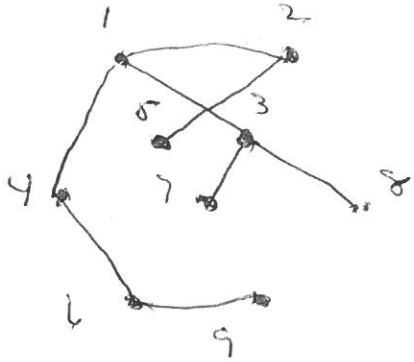
We look at examples



Steps	U	$bf(u)=$	$D(u)$	F	T
$l=1$	u_1	1	0	ϕ	(U,F)
$l=2$	u_2	2	1	u_1-u_2	(U,F)
$l=3$	u_3	3	1	u_1-u_2, u_1-u_3	(U,F)
$l=4$	u_4	$bf(u)=4$	1	u_1-u_2, u_1-u_3, u_1-u_4	(U,F)
5	u_5	5	2	$F \cup (u_2-u_5)$	(U,F)
6	u_6	6	2	$F \cup (u_3-u_6)$	(U,F)
7	u_7	7	2	$F \cup (u_4-u_7)$	(U,F)
8	u_8	8	3	$F \cup (u_5-u_8)$	(U,F)
9	u_9	9	3	$F \cup (u_6-u_9)$	(U,F)
10	u_{10}	10	3	$F \cup (u_7-u_{10})$	(U,F)
11	u_{11}	11	4	$F \cup (u_{10}-u_{11})$	(U,F)

Spanning Tree





$d(1)=0$ $\quad$ $d(4)=1$ $\quad$ $d(7)=2$

$d(2)=1$ $\quad$ $d(5)=2$ $\quad$ $d(8)=2$

$d(3)=1$ $\quad$ $d(6)=2$ $\quad$ $d(9)=3$

Spanning Tree

Problem. Use breadth First algorithm to find a spanning tree for G whose verticles are the vertices of a cube and whose edges are the edges of the cube

Make a chart like in our first example

Problem. Find all trees of order 5

Some Solutions

40. Let $|G| = n$. If G has $t \geq \frac{(n-1)(n-2)}{2} + 2$ edges, then G has a Hamilton cycle.

Proof. We will use the Ore condition.

Suppose $x, y \in G$ but there is no edge between them and $\deg x + \deg y = s < n$

Let G' be the subgraph of G obtained by removing and all edges that contain them. There are s of them since $x-y$ is not an edge.

The number of edges in G' is $t-s$

$$> \frac{(n-1)(n-2)}{2} + 2 - n = \frac{n^2 - 3n + 2}{2} - n + 2$$

$$= \frac{n^2 - 5n + 6}{2} = \frac{(n-3)(n-2)}{2} = \binom{n-2}{2}$$

But $|G'| = n-2$. It can not contain more than $\binom{n-2}{2}$ edges. This contradiction gives $\deg x + \deg y \geq n$

So by the Ore condition, G has a Hamilton cycle

20. $|G| = n$ If G has $r \geq \frac{(n-1)(n-2)}{2} + 1$

edges, then G is connected.

Suppose not. Then $G = S \cup T$, $S \cap T = \emptyset$ and there are no edges between S and T. Suppose $|S| = k$, then $|T| = n-k$

S has less than or equal to $\binom{k}{2}$ edges

T has less than or equal to $\binom{n-k}{2}$ edges

The Total number of edges in

$G = S \cup T$ is $\leq \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} =$

$\frac{k^2 - k + n^2 - nk - n - kn + k^2 + k}{2} =$

$\frac{1}{2}[n^2 - 2nk + 2k^2 - n] = f(k)$ (n is fixed)

The minimum of ~~Thi~~ $f(k)$ is when $k = \frac{1}{2}n$

$f(k)$ is a parabola opening up and $\frac{1}{2}n$ is between end points $1 \leq k \leq n-1$

The maximum occus at The endpoints

Check $f(k) = f(n-1) = \frac{(n-2)(n-1)}{2} < r$

So there is no way To pick S and T and get G with the condition Hence G is connected

Problem Set 8

I Let G be a graph whose verticies are the verticies of a cube and whose edges are the edges of the cube. In a and b answer yes or no and ~~I~~ Explain your answer

a. Does G have a closed Eulerian Trail

b. Does G have a Hamilton cycle

c. Construct a breadth first spanning Tree

II, Let G be a complete graph K_n

a. For what n ~~d~~ is there a closed Eulerian Trail

b For what n is their a Hamiltonian cycle

c. Construct a spanning Tree What do you notice

d For K_7 construct a ~~k~~ closed Eulerian Train

III Let G be a graph with verticies x and y be the only verticies of odd degree. Let G' be G with edge x-y added. Prove G is connected if and only if G' is connected