

Polya Theory

POLYTA THEORY

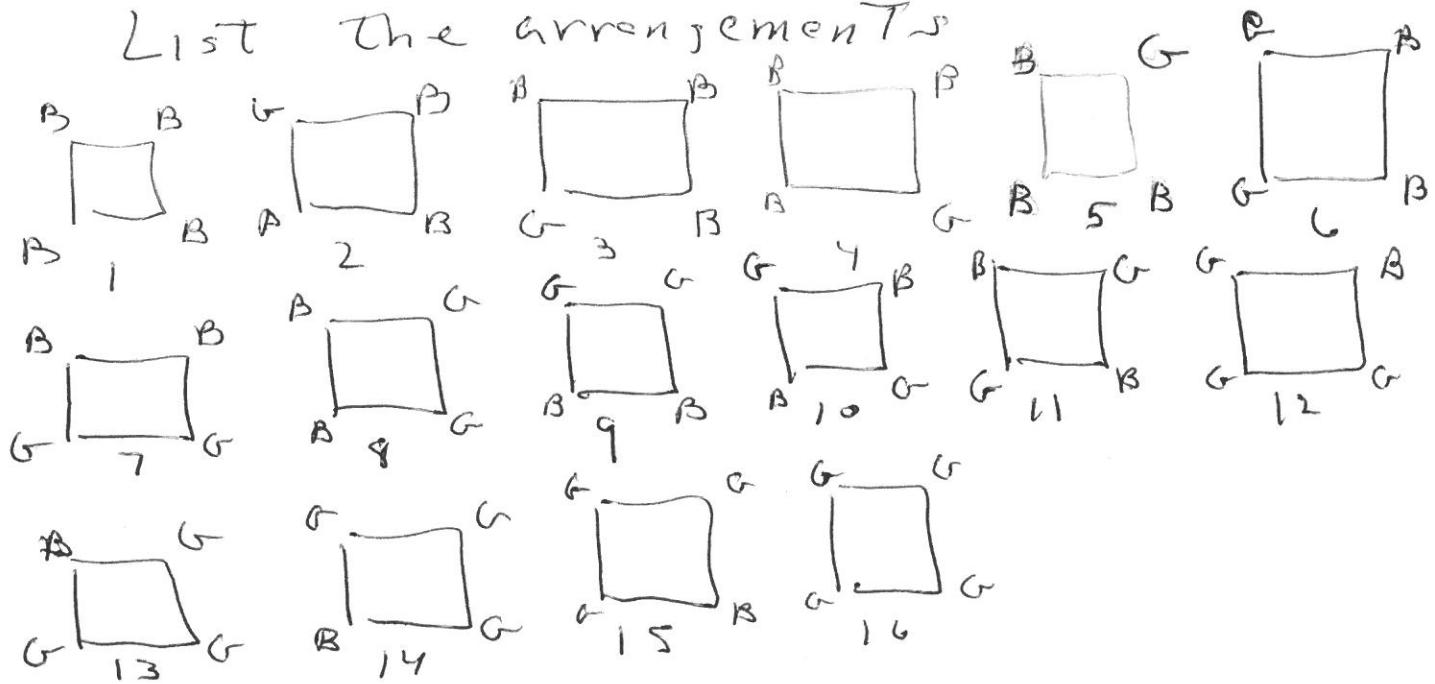
An example

Polya theory is a counting method that uses group theory and combinatorics

We begin with an example

Consider necklaces with 4 beads which are either blue or green

List the arrangements



X

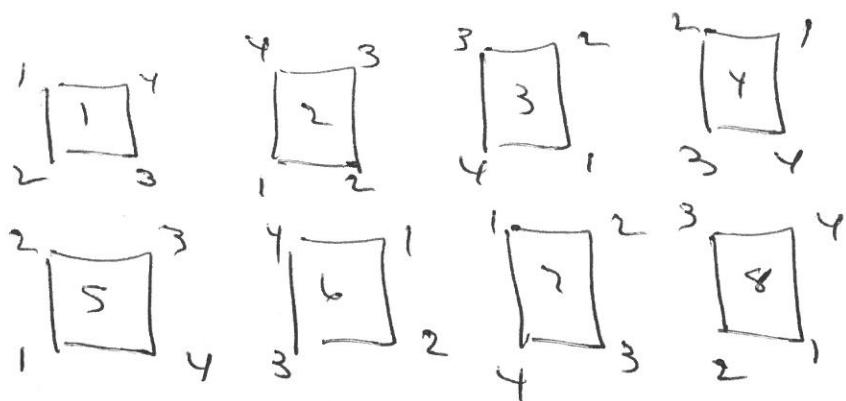
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Not all of these necklaces are really different. For instance 2 and 3 are the same, a simple rotation takes 2 into 3. We always need to decide what makes 2 necklaces the same; in fact, we define it. Here we will say they are the same if there is a rotation or a reflection which takes one into the other. The collection of all rotations and reflections used in the problem forms a group G under composition. This will always be the case. We can also consider these motions to act on a square



which does not have the colors on it. The advantage is that we have fewer squares:

3



S

Each motion is actually a permutation
on the squares' vertices

$$g_1 = \text{identity} \quad (1)(2)(3)(4)$$

$$g_2 = 90^\circ \text{ rotation} \quad (1234)$$

$$g_3 = 180^\circ \text{ rotation} \quad (13)(24)$$

$$g_4 = 270^\circ \text{ rotation} \quad (1432)$$

$$g_5 = \text{refl. } \square \quad (12)(34)$$

$$g_6 = \text{refl. } \square \quad (14)(23)$$

$$g_7 = \text{refl. } \square \quad (1)(3)(24)$$

$$g_8 = \text{refl. } \square \quad (13)(2)(4)$$

Def Let G be a group and Y be a set. G acts on Y if each $g \in G$ is a permutation of Y with product.

that has $(hg)g^{-1} = h(g(g)) \quad h, g \in G$
 $e(y) = y \quad e = \text{identity}$

In our example, G is the dihedral group of order 8.

⁴ Lemma. Suppose G acts on Y .

Define the relation on Y , $x \sim y$ if there exists $g \in G$ such that $g(x) = y$. \sim is an equivalence relation and the equivalence classes are called orbits.

Notice that G also acts on the original set S . For example the

90° rotation gives $(1)(2\ 3\ 4\ 5)(6\ 7\ 8\ 9)(10\ 11)(12\ 13\ 14\ 15)(16)$

Notice that the permutations are much longer

Dot Suppose G acts on Y and $g \in G$. Let $\text{Fix}(g) = \{y \in Y / g(y) = y\}$

We compute $\text{Fix}(g)$ when G acts on X

In our example

$$g \in G \quad \text{Fix}(g)$$

$$|\text{Fix}(g)|$$

$$\begin{matrix} \text{Tol}_g \\ |\text{Fix}(g)| \end{matrix}$$

$$g_1 \quad X$$

$$16$$

$$11$$

$$g_2 \quad 1, 16$$

$$2$$

$$48$$

$$g_3 \quad 1, 10, 11, 16$$

$$2$$

$$g_4 \quad 1, 16$$

$$4$$

$$g_5 \quad 1, 6, 8, 16$$

$$4$$

$$g_6 \quad 1, 2, 9, 16$$

$$8$$

$$g_7 \quad 1, 2, 4, 10, 11, 13, 15, 16$$

$$8$$

$$g_8 \quad 1, 3, 5, 10, 11, 12, 14, 16$$

Def Contrasting $\text{Fix}(g)$ we have

for $y \in Y$, $S_{\text{stab}}(y) = \{g \in G / g(y) = y\}$

Continue the example

$x \in X$	$S_{\text{stab}}(x)$	$ S_{\text{stab}}(x) $
1	G	8
2	g_1, g_7	2
3	g_1, g_8	2
4	g_1, g_7	2
5	g_1, g_8	2
6	g_1, g_5	2
7	g_1, g_4	2
8	g_1, g_5	2
9	g_1, g_4	4
10	g_1, g_3, g_2, g_8	4
11	g_1, g_3, g_7, g_8	4
12	g_1, g_8	2
13	g_1, g_7	2
14	g_1, g_8	2
15	g_1, g_7	2
16	G	8

$$\text{Total} = 48$$

Notice that the totals in both examples are the same. This is not an accident.

Lemma Suppose G acts on Y . Then

$$\sum_{g \in G} |\text{Fix}(g)| = \sum_{y \in Y} |\text{Stab}(y)|$$

Proof Let $T = \{(g, y) / g \in G, y \in Y, g(y) = y\}$

Count going down the first column collecting for similar g 's, the count down the second column counting for similar y 's

The result are $|T|$ and equal

the terms in the equation,

Def Let G act on Y . For each $y \in Y$

Let $\text{Orb}(y) = \{x \in Y / x = gy \text{ for some } g \in G\}$

For each $y \in Y$, suppose G acts on Y .

Lemma Suppose G acts on Y .

$|\text{Orb}(y)| = |\text{Stab}(y)|$

Proof This follows from Lagrange's theorem

as $|\text{Orb}(y)| = \text{number of left cosets}$

of the subgroup $\text{Stab}(y)$

If $g, h \in \text{Same left coset}$, then

If $g, h \in \text{Same left coset}$, then $g = hs$ for some $s \in \text{Stab}(y)$, $g(y) = h(s(y))^{-1}h(y)$

so g and h take s to the same element in the orbit of s . On the other hand, suppose $g(y) = h(y)$. Then $y = h^{-1}g(y)$ and $h^{-1}g \in \text{Stab}(y)$, $h^{-1}g = s$ $\Rightarrow h = gs$ so g and h are in the same left coset. So there is a one to one correspondence between the left cosets of $\text{Stab}(y)$ and the elements in the orbit of y , so they have the same order. Thus $|G| = |\text{Stab}(y)|$ left cosets of $\text{Stab}(y) = |\text{Stab}(y)| |\text{Orbit}(y)|$

Burnside's Thm.

Let G act on Y . The number of orbits in Y is $\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$

Proof: From the last two results,

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \frac{1}{|G|} \sum_{y \in Y} |\text{Stab}(y)| = \sum_{y \in Y} \frac{1}{|\text{Orb}(y)|}$$

Suppose there are t orbits;

$\forall y \in Y$ For $x, y \in O_i$, $|\text{Orb}(x)| = |\text{Orb}(y)| = |O_i|$

$O_1 - O_t$.

$$\text{Hence } \sum_{y \in Y} \frac{1}{|\text{Orb}(y)|} = \underbrace{\frac{1}{|O_1|} + \dots + \frac{1}{|O_t|}}_{\text{of them}} = 1 \quad *$$

$$\text{Hence } \sum_{y \in Y} \frac{1}{|\text{Orb}(y)|} = s \text{ since there are } s \text{ of } *$$

Ex. In our example $\sum |\text{Fix}(g)| = 48$

$$\text{So } \frac{1}{|\mathcal{G}|} \sum |\text{Fix}(g)| = \frac{48}{8} = 6$$

There are 6 different necklaces

Problems. Repeat the example for

$$n=3, n=5, n=6$$

The Cycle Index

In the last section we managed to find the number of necklaces by using the set X . In a larger example, this would not be feasible. We will use the smaller set S to work on such problems. In S we have



For the motion that is a reflection through the 2-4 diagonal, if the associated element in X is to be in $\text{Fix}(g)$, the 1 and 3 must be assigned the same color but 2 and 4 can have different colors. This is because 1 and 3 are

in the same cycle of the permutation
 $(13)(2)(4)$. That said, in this permutation
we get to make 3 color choices, one

for each cycle. The same can be

said for any permutation. we have
2 colors and 3 cycles for this permutation
hence $2 \cdot 2 \cdot 2 = 2^3$ choices. Using this

idea for each of the permutations

$$\text{on } S, \sum_{g \in S} |F_{\chi}(g)| = \sum_{g \in S} 2^{k_g} \text{ where } k_g$$

is the number of cycles in g acting

on S . In our example

$$\sum |F_{\chi}(g)| = 2^4 + 2^1 + 2^2 + 2^1 + 2^2 + 2^3 + 2^3 = 48$$

If we have 5 colors,

$$\sum |F_{\chi}(g)| = 5^4 + 5^1 + 5^2 + 5^1 + 5^2 + 5^3 + 5^3 = 960$$

and the number of patterns is $\frac{960}{8} = 120$

$$= \frac{1}{120!} \sum |F_{\chi}(g)|.$$

We construct a polynomial from
these ideas. Represent a cycle of

length k by x_k and represent a
permutation by the product of the x_k

that comes from the cycle structure

Ex Continued

gen one S

(12)(3)(4)

(1234)

(12)(24)

(1432)

(12)(34)

(14)(23)

(1)(24)(3)

(13)(2)(4)

Polynomial

x_1^4

x_4

x_2^2

x_3^4

x_2^2

$x_2 x_1^2$

$x_2 x_1^2$

The polynomial is the sum of these monomials

$$f(x_1, x_2, x_3, x_4) = \frac{1}{16} (x_1^4 + 2x_4 + 3x_2^2 + 2x_3^2 x_2)$$

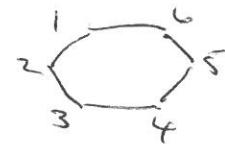
If there are 2 colors

$$f(2, 2, 2, 2) = \frac{1}{8} [16 + 4 + 12 + 16] = 6$$

If there are 5 colors

$$f(5, 5, 5, 5) = \frac{1}{8} [625 + 10 + 75 + 250] = 120$$

different necklaces.
The polynomial is called the
CYCLE INDEX for the problem



There are 12 elements in G
6 rotations and 6 reflections

g	Permutation	Monomial
0 rotation	$(1)(2)(3)(4)(5)(6)$	x_1^6
60 rotation	(123456)	x_6^6
120 "	$(135)(246)$	x_3^2
180 "	$(14)(25)(36)$	x_2^3
240 "	$(153)(264)$	x_3^2
300 "	(165432)	x_6
2-5 reflection	$(2)(5)(13)(24)$	$x_1^2 x_2^2$
3-6 reflection	$(3)(6)(24)(15)$	$x_1^2 x_2^2$
1-4 "	$(1)(4)(35)(26)$	$x_1^2 x_2^2$
	$(25)(16)(34)$	x_2^3
	$(12)(36)(45)$	x_2^3
	$(65)(14)(23)$	x_2^3

Cycle index

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{12} \left[x_1^6 + 2x_6 + 2x_3^2 + 4x_2^3 + 2x_1^2 x_2^2 \right]$$

2 colors $f(2, 2, 2, 2, 2, 2) = 13$

5 colors $f(5, 5, 5, 5, 5, 5) = 1505$

There is another way to view our running example, from group theory. In the example we considered the motions of the square, 4 rotations and 4 reflections. The resulting group is the dihedral group D_4 . We also showed its elements by permutations. We eventually computed the cycle index. This is the cycle index for D_4 .

For each n , there is a similar process. Acting on a regular polygon with n vertices (and sides), the group of motions consists of n rotations and n reflections. We can list them as permutations and obtain the cycle index for the group, D_n . We can then answer coloring questions as we did for D_4 .

Problem: Compute the cycle index for D_3 and D_5

The Pattern Inventory

Going back to 2 colors and the square, we can ask how many of the necklaces have exactly 2 colored beads of each type. Each permutation consists of cycles, made up of vertices. For a necklace to be fixed, in each cycle the vertices must have either all blue or all green. In the 180° rotation there are 2 cycles each of length 2. The first can be colored with 2 blues or 2 greens, we list as

$$B^2 + G^2$$

The second has the same colorings $B^2 + G^2$. The product $(B^2 + G^2)^2 = B^4 + 2B^2G^2 + G^4$

The contribution to substitution into
the cycle index is

$$(B^2 + G^2)^2 = B^4 + 2B^2G^2 + G^4$$

We do this for each permutation
in D_4 , or substitute into the
cycle index:

$$f(x_1, x_2, x_3, x_4) = \frac{1}{8} [x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2]$$

$$f(B+G, B^2+G^2, B^3+G^3, B^4+G^4) =$$

$$\frac{1}{8} [(B+G)^4 + 2(B^4+G^4) + 3(B^2+G^2)^2 +$$

$$2(B+G)^2(B^2+G^2)] =$$

$$B^4 + B^3G + 2B^2G^2 + BG^3 + G^4$$

The coefficient of each term gives
the number of necklaces with
the designated colors. This expansion's
terms are the pattern inventory

All this is summarized in Polya's
theorem

Ex. Suppose we ask for the number of 4 bead necklaces with 3 colors, blue, green, red

We use the cycle index for D_4

$$f(x_1, x_2, x_3, x_4) = \frac{1}{8} (x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2)$$

To compute the pattern inventory:

$$\begin{aligned} f(B+G+R, B^2+G^2+R^2, B^3+G^3+R^3, B^4+G^4+R^4) \\ = \frac{1}{8} \left[(B+G+R)^4 + 2(B^4+G^4+R^4) + 3(B^2+G^2+R^2)^2 \right. \\ \left. + 2(B+G+R)^2(B^2+G^2+R^2) \right] = \\ B^4 + B^3G + 2B^2G^2 + BG^3 + \cancel{G^4} + \\ + 2B^4 + B^3R + 2B^2R^2 + BR^3 + \\ G^4 + G^3R + 2G^2R^2 + GR^3 + \\ 2B^2GR + 2BG^2R + 2BGR^2 \end{aligned}$$

The inventory tells the there are 2 necklaces with 1 B, 2 G and 1 R

It also answers how many are there with at least 2 B: $1 + 1 + 2 + 1 + 2 + 2 = 9$
by reading off coefficients.

Theorem (Polya). After substituting colors into the cycle index and reducing, the coefficient in front of a monomial of colors is the number of patterns that have each color appearing the number of times that is the exponent of the color.

Problem. A domino is 2 squares, one on top of the other with from 1 to 6 dots:



2 Dominoes are the same if one can be rotated to the other.

How many dominoes are there?

Problem. Same as the last problem but now suppose the back of the dominoe also has dots. We can now also turn the dominoe over. List the permutations for the motions, find the cycle index and how many dominoes are there.

Problem. A pentagon has a color at each vertex. There are 2 colors.

Find the cycle index

Find how many ~~color~~ necklaces are there

Find how many necklaces have all 3 colors.

Problem An ornament looks like,



It is anchored at the top but can rotate at the 3 other connections. Find all permutations and the cycle index. Suppose each connection (except the top) has a blue or green bell. How many ornaments are there?

The Cycle Index for S_n

Finding the cycle index for S_n is the same as for D_n . We find the different cycle structures, turn them into monomials, find how many there are of each type and write down the polynomial.

Ex. S_5 . To find the different cycle structures, we find the partitions of $n=5$. There are the ways that positive integers can add to $n=5$.

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$$

For each of these partitions, there are permutations with cycles of these types

Partition	Permutation	Monomial	Number
5	(12345)	x_5	24
4+1	(1234)(5)	$x_4 x_1$	30
3+2	(123)(45)	$x_3 x_2$	20
3+1+1	(123)(4)(5)	$x_3 x_1^2$	20
2+2+1	(1~)(34)(5)	$x_2^2 x_1$	15
2+1+1+1	(12)(3)(4)(5)	$x_2 x_1^3$	10
1+1+1+1+1	(1)(2)(3)(4)(5)	x_1^5	1

To compute the numbers of each type:

List by partition

5 choose numbers $\binom{5}{5}$ ways

List as circular permutation

$$\frac{5!}{5} = 4!$$

$$\text{Ans: } 4! = 24$$

(4+1) choose number $\binom{5}{4} = 5$

List as circular permutation $\frac{4!}{4} = 3!$

$$\text{Ans: } 5 \cdot 3! = 30$$

$$3+2 \quad \binom{5}{3} = 10$$

$$\frac{3!}{3} = 2!$$

$$\text{Ans: } 10 \cdot 2! = 20$$

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$$3+1+(\frac{5}{3}) = 10 \quad \frac{3!}{3} = 2 \quad \text{Ans} = 20$$

$$(\frac{3}{1}) \cdot \frac{1}{2!} = 1$$

The $2!$ appears because there is a number repeated twice in $3+1+1$

$$2+2+1 \quad (\frac{5}{2}) = 10$$

$$\frac{2!}{2} = 1$$

$$(\frac{3}{2}) = 3$$

$$\frac{2!}{2} = 1$$

Since 2 is repeated twice, divide by $2!$

$$\text{Ans} = 10 \cdot 3 \cdot \frac{1}{2} = 15$$

$$2+1+1+1$$

$$(\frac{5}{2}) = 10$$

$$\frac{2!}{2} = 1$$

Trailing ones always give a contribution of 1

$$\text{Ans} = 10$$

$$1+1+1+1+1$$

$$\text{Ans} = 1$$

List these in the table on the previous page

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Check that the last column
adds to $n! = 5! = 120$

Cycle index

$$\frac{1}{120} \left[24x_5 + 30x_4x_1 + 20x_3x_2 + 20x_3x_1^2 + 15x_2^2x_1 + 10x_2x_1^3 + x_1^5 \right]$$

A general polynomial formula exists
for S_n . We will just refer
to the text.

As an application, suppose a donut shop has 3 kinds of donuts in unlimited supply. Suppose a customer wants 5 donuts. In how many ways can the order be filled?

Let x_1, x_2 and x_3 stand for the number of donut type 1, 2 and 3 in an order. Then $x_1 + x_2 + x_3 = 5$ and we want to know the number of triples of non-negative integers add up to 5. Suppose y_1, y_2, y_3, y_4 and y_5 are the order the donuts are picked. For a given x_1, x_2, x_3 , there is no difference in the order of picking them. Given another order z_1, z_2, z_3, z_4, z_5 . There is a permutation of the first 1st to the second, an element of S_5 . Hence let S_5 act on y_1, y_2, y_3, y_4, y_5 . The cycle index is g , we have seen $\frac{1}{120} (24x_5^2 + 30x_4x_1 + 20x_3x_2 + 20x_2x_3 + 15x_2^2x_1 + 10x_2x_1^3 + x_1^5) = f(x_1, x_2, x_3, x_4, x_5)$. We are looking for the number of ways of coloring the choice order of choosing using the 3 colors (donut types).

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The answer is

$$f(3,3,3,3,3) = \frac{1}{120} [72 + 270 + 180 + 540 + \\ 405 + 810 + 243] = 21$$

Problem. Given 10 billiard balls
number 1 to 10. Arrange them
in a triangle

Two arrangements are the same if
there each ball has the same
neighbors in the two arrangements?
What is the group of motions?
What is the cycle index for the group?
How many different arrangements
are there?