

Chapter 4

Combinatorics

CHROMATIC POLYNOMIAL

# 1 Chromatic Polynomial

Let  $G=(V,E)$ ,  $|G|=n$ . Given  $k$ , how many  $k$  colorings exist. Call this  $P_G(k)$ .

If  $k < \chi(G)$ , then none exist, so  $P_G(k)=0$ .

Ex. Let  $G$  be a tree. Pick any vertex,  $v$ , it can be colored by any of the  $k$  colors. Each vertex adjacent to  $v$ .

They can be colored by  $k-1$  colors each.

Following this pattern each of the next adjacent vertices can receive  $k-1$  colors. and so on. Hence  $P_G(k) = k(k-1)^{n-1}$

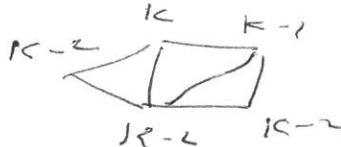
Ex. Let  $G$  be  $N_n$  the null graph with  $n$  vertices. Clearly  $P_G(N_n) = k^n$

Ex.  $G = K_n$ . A first vertex can have  $k$  colors, a second has  $k-1$  choices. Since each new vertex is adjacent to all the previous ones, it can not get one of their colors. If  $v_1, \dots, v_n$  are the vertices:

TABLE		Number of colors
$v_1$	$k$	
$v_2$	$k-1$	
$v_3$	$k-2$	
$v_4$	$k-3$	
		$v_n$ gets $k-n+1$ choices

(Hence  $P_G(k) = k(k-1)\dots(k-n+1) =$

The number of  $n$  permutations of  $k$  elements

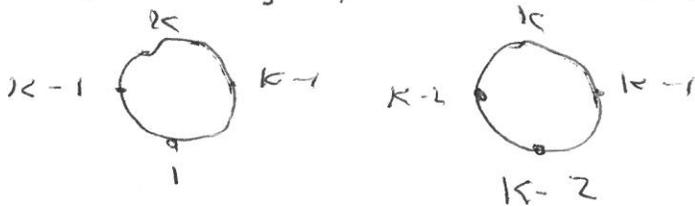
Ex   $P_G(k) = k(k-1)(k-2)^3$

When cycles exist (ie not Trees) These assignments can get tricky



What goes in? It depends

If the object at top and bottom are different,  $? = k-2$  If not  $? = k-1$



$$P_G(k) = k(k-1)^2 + k(k-1)(k-2)^2$$

As you can guess, if this simple example requires cases, things can get complicated

By the way, in our examples,  $P_G(k)$  is a polynomial. This is always the case. It is called the chromatic polynomial for  $G$  and its values for each  $k$  gives the number of colorings

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There is an algorithm to compute  $P_G(k)$ . Each step has 2 parts, so it can be quite lengthy. The idea is to reduce the graph to a collection of null graphs, with varying number of vertices which number gives a monomial in  $P_G(k)$ . After all is reduced there might be, say, 5 null graphs with 4 vertices. This contributes  $k^4$ . If there are 5 of them, we get  $5k^4$ . It is actually  $\pm 5k^4$  and a miracle in this algorithm is that some of the  $k^4$ 's ~~had~~ don't have different signs. They are all either  $+$  or  $-$ .  $P_G(k)$  shows many properties of the graph as we will see.

### Algorithm for $P_G(k)$

We do this algorithm until there are no edges left!

1. Pick  $x$  and  $y$  such  $x-y \in E$  □

2. Remove  $x-y$  to get a new graph  $G_1$  □

3.  $G_1$  has 2 possibilities, either the  $x$  and  $y$  are colored different or colored the same (In  $G_1$  with  $G$ )

4. Let  $C(K)$  = graphs in  $G_1$ ,  $x$  and  $y$  different  
 $C(K)$  is in 1-1 correspondence with  
 graphs coloring in  $G$

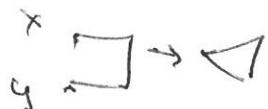
5. Let  $D(K)$  = graphs in  $G_1$ ,  $x$  and  $y$  colored  
 the same

$$\text{Clearly } P_{G_1}(K) = C(K) + D(K)$$

$$P_G(K) = P_{G_1}(K) + D(K)$$

$$\rightarrow P_G(K) = P_{G_1}(K) - D(K)$$

6. In The graphs in  $D(K)$ , fuse  $x$  and  $y$   
 with all edges adjoining

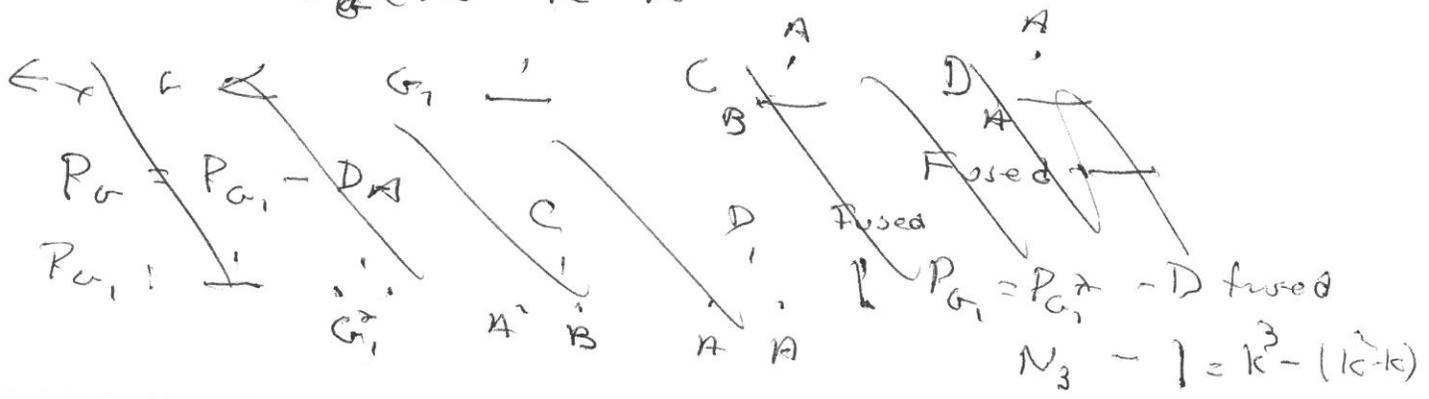


Notice the number of edges in  
 both  $P_{G_1}(K)$  and  $D(K)$  are less than  
 those in  $P_G(K)$ . This means if  
 we repeat this enough times  
 (the number of edges to start  
 with would be an upper bound)  
 we get null graphs, just what  
 we want. So  $P_G(K)$  = combination of  
 monomials obtained from these null  
 graphs. Also note that the number  
 of vertices does not drop in  $P_{G_1}(K)$   
 but drops by 1 in  $D(K)$   
 Suppose we start with 7 vertices and  
 in one of these sequences end with

a null graph with 4 vertices  
 It has taken 2-4 sign changes  
 to go from 7 to 4 so the  
 sign on the monomial  $k^4$  will  
 be negative when we plug it  
 back into this huge expanded  
 expression. And it happens each  
 time we go from 7 vertices to 4  
 Ah In general, this always happens  
 The monomial  $x^s$  in the final expression  
 has coefficient = the number of  
 null graphs with 4 vertices and  
 its sign is the difference  $t-s$  when  
 we have  $t$  vertices in  $G$ .

Ex  $G \rightarrow G_1 \dots C \begin{matrix} \cdot \\ A \end{matrix} \begin{matrix} \cdot \\ B \end{matrix} D \begin{matrix} \cdot \\ A \end{matrix} \begin{matrix} \cdot \\ A \end{matrix}$   
 $G_1 = C + D$   
 $G_2 = G_1 + D$   
 $P_G(k) = P_{G_1}(k) - D(k)$

we are already there  
 $G_1$  null graph 2 vertices  $k^2$   
 $D$  fused null graph 1 vertex  
 $P_G(k) = k^2 - k$



Ex  $G: \angle$

$$\angle = C + D = \frac{A}{B} + \frac{A}{A} = G + \frac{A}{D_{\text{fused}}}$$

$$\rightarrow G = \angle - \frac{A}{D_{\text{fused}}}$$

$$\angle = \frac{A}{A} + \frac{A}{B} = \frac{A}{A} + \frac{A}{B}$$

$$\rightarrow \angle = \frac{A}{A} + \frac{A}{B} - \frac{A}{B} \rightarrow K^3 - K^2$$

$$\angle = \frac{A}{A} + \frac{A}{B} + \frac{A}{B} = \frac{A}{A} + \frac{A}{B}$$

$$\angle = \frac{A}{A} - \frac{A}{B} = K^2 - K$$

So

$$G = \angle - \frac{A}{B} = (K^3 - K^2) - (K^2 - K)$$

$$P_G(K) = K^3 - 2K^2 + K$$

$$P_G(2) = 8 - 8 + 2 = 2 \rightarrow \begin{matrix} A \\ \swarrow \quad \searrow \\ B \quad A \end{matrix}, \begin{matrix} B \\ \swarrow \quad \searrow \\ A \quad B \end{matrix}$$

Notice! there are 3 vertices, the

highest exponent

there are 2 edges, the

coefficient (+ or -) of  $K^{n-1} = K^2$

the constant is 0

the term  $K$  appears

which says  $G$  is connected.

7 This gets complicated fast.

A computer algebra package would have the algorithm available

What the algorithm shows us is we get a polynomial after some steps. ALSO

1. The degree of  $P_G(k) =$  number of vertices
2. The constant in  $P_G(k)$  is 0 since we have added null graph  $P_G(k)$ 's and they are not constant
3. In connected graphs, one of the null graphs has one vertex so  $k$  appears as a first degree term. The coefficient is the number of null graphs with one vertex and the sign is obtained from  $(-1)^{|V|-1} =$   
+1 if  $|V|-1$  is even  
-1 if  $|V|-1$  is odd
4. If  $G$  has just 2 connected components  $G_1$  and  $G_2 \rightarrow P_G = P_{G_1} P_{G_2}$ . This is shown by doing the ~~reduction~~ reduction algorithm first on  $G_1$ , then  $G_2$

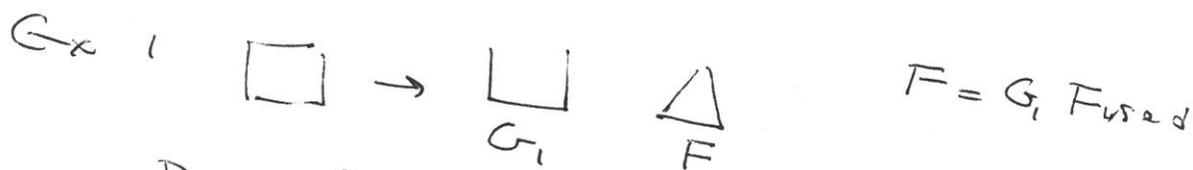
This requires some details, but not many. In this case  $P_G(k)$  will have a  $k^2$  term but no  $k$  term (check what happens when you multiply 2 polynomials, each with a  $k$  term but no constant term) More generally,  $G$  has  $s$  connected components iff  $k^s$  is the smallest degree in  $P_G(k)$

Also, the coefficient of  $k^{n-1}$  is  ~~$-m$  where  $m$  is the number of~~  $-m$  where  $m$  is the number of edges in the graph. The algorithm and induction shows this (Induction on the number of edges):

If  $G$  has  $m$  edges  
 $G_1$  has  $m-1$  edges  
 $\rightarrow P_{G_1}(k) = k^n - (m-1)k^{n-1} + \dots$   
 $D_{\text{fused}} = k^{n-1} \dots$  since one vertex is gone  
 $\rightarrow P_G = P_{G_1} - D_{\text{fused}} = k^n - m k^{n-1} + \dots$   
 and the result appears

1  
Notice also in the examples  
that the signs alternate.  
This is explained. This is  
explained by signs alternating  
as we eliminate vertices in  
the algorithm in the argument  
that we used

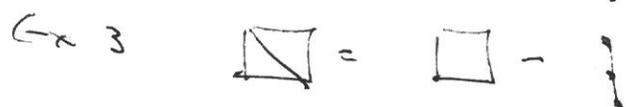
We will compute some chromatic polynomials by just doing the first step to reduce down to known graphs



$$P_G = P_{G_1} - P_F = k(k-1)^3 - k(k-1)(k-2) = k^4 - 4k^3 + 6k^2 - 3k$$



$$P_G = k^4 - 4k^3 + 6k^2 - 3k - (k(k-1)^2) = k(k-1)^4 - (k^4 - 4k^3 + 6k^2 - 3k)$$



$$P_G = k^4 - 4k^3 + 6k^2 - 3k - k(k-1)^2$$



$G_1$  is  $K_4$  with an extra vertex and 2 edges

$$P_{G_1} = k(k-1)(k-2)(k-3)(k-2)$$

$$P_F = k(k-1)(k-2)(k-3)$$

$$P_G = P_{G_1} - P_F = k(k-1)(k-2)(k-3)(k-2-1)$$

Ex 5 If  $G$  is an  $n$ -cycle, then

$$P_G = (k-1)^n + (-1)^n (k-1)$$

Induct on  $n$

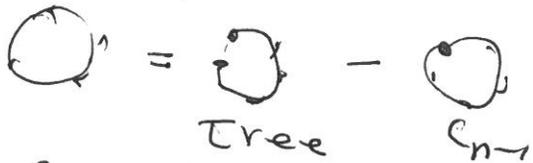


$$P_G = k(k-1)^2 - k(k-1) = (k-1)^2 - (k-1)$$

Assume for  $G = C_{n-1}$

$$P_G = (k-1)^{n-1} + (-1)^{n-1} (k-1)$$

For  $C_n$


$$C_n = \text{Tree} - C_{n-1}$$

$$\begin{aligned} P_G &= k(k-1)^{n-1} - ((k-1)^{n-1} + (-1)^{n-1} (k-1)) \\ &= (k-1)^n + (-1)^n (k-1) \end{aligned}$$