

CHAPTER 2
COMBINATORICS

Counting Graphs

GRAPH THEORY

Def. A graph G is a set V of vertices and a set E of pairs (x, y) of elements of V . The elements of E are called edges and, here, the edges are undirected meaning $(x, y) = (y, x)$

Ex $G \models V = \{1, 2, 3, 4\} \quad E = \{(1, 2), (1, 3), (2, 4)\}$

When V only has a few elements we draw G



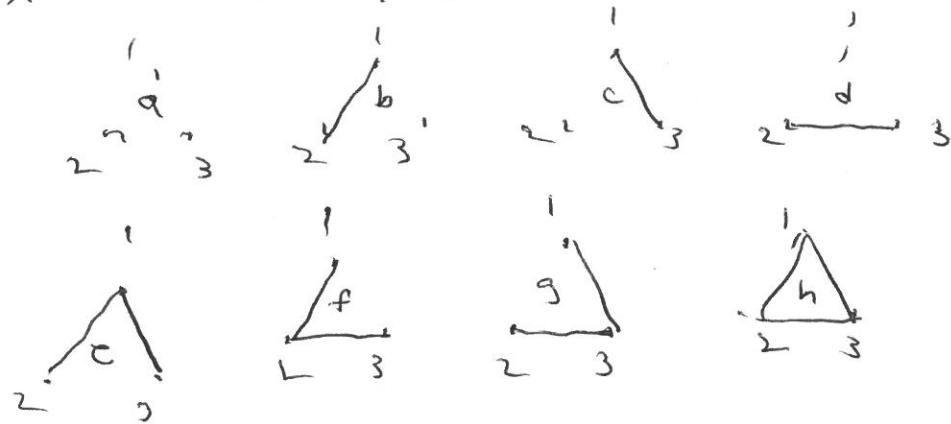
We will use the notation $a-b$ to show $(a, b) \in E$. Then $a-b-c-d$ means $(a, b), (b, c), (c, d) \in E$ and say there is a path from a to d . G is connected if for each $x, y \in V$, they are connected by a path

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Our first study is to find

the number of graphs using
n vertices in a certain sense

$$\text{Ex } V = \{1, 2, 3\}$$



We say graphs are equivalent if there is a bijection on the vertices which aligns corresponding edges

In our example it is easy to see that all 3 graphs with 1 edge are equivalent as are all 3 graphs with 2 edges

3 The set of all the bijections
are the permutations on the
n vertices; S_n

We will now contrast this problem
with the bead problem at the
beginning of the Polya Theory
investigation.

In Contrast

Bead Problem

S Uncolored Positions

X All necklaces

R The colors

G D_n

Graph Problem

All Edges

All graphs

$\{0, 1\}$

S_n

In both cases we want the
number of non-equivalent elements
in X and we use S and G
to compute a cycle index and then

4 Substitute the number of elements in R into the cycle index polynomial to find the answer

Ex. Continued. Here $n=3$. So S_3 acts on the set of edges

$$e_1 = (1, 2) \quad e_2 = (1, 3) \quad e_3 = (2, 3)$$

We take an element of each cycle type in S_3 and also count the number of elements of that cycle type and record it in the last column of the following table

Element in S_3	Action on S	Action on X	Number of permutations of this cycle type
$(1)(2)(3)$	$(e_1)(e_2)(e_3)$	Identity	1
$(12)(3)$	$(e_1)(e_2e_3)$	$(cde)(efg)$	3
(123)	$(e_1e_3e_2)$	$(badc)(efg)$	2

The monomials gotten from column 2 are
 x_1^3, x_1x_2, x_3

The cyclic index is

$$f(x_1, x_2, x_3) = \frac{1}{|S_3|} (x_1^3 + 3x_1x_2 + 2x_3)$$

so

$$+(2, 2, 2) = \frac{1}{6} (8 + 12 + 4) = 4$$

There are 4 non-equivalent graphs on 3 vertices. This result is obvious from the

Example $V = \{1, 2, 3, 4, 5\}$

$$S = E = \{e_1 = (1, 2), e_2 = (1, 3), e_3 = (1, 4), e_4 = (1, 5)$$

$$e_5 = (2, 3), e_6 = (2, 4), e_7 = (2, 5)$$

$$e_8 = (3, 4), e_9 = (3, 5), e_{10} = (4, 5)\}$$

where we assign an e_i to each edge $\in S$, and the number of graphs is 2^{10} since there are 10 edges each with 2 possibilities of being in a given graph.

To construct the cycle index
 we will use the number of permutations
 of each cycle type which we now
 discuss

(12345) : Choose all 5 numbers in V $\binom{5}{5}$

Then arrange using circular permutations

$$\frac{5!}{5} = 4! \quad \text{Number} = \binom{5}{5} 4! = 4! = 24$$

$(1234)(5)$ Choose 4 out of 5 for the first
 cycle $\binom{5}{4}$. Use circular permutations

$$\frac{4!}{4} = 3! \quad \text{Total} = \binom{5}{4} 3! = 30$$

$(123)(45)$ Choose 3 out of 5, $\binom{5}{3}$. Circular
 permutations $\frac{3!}{3} = 2!$. Then choose
 2 out of 2 $\binom{2}{2}$ and circular
 permutations $\frac{2!}{2} = 1$

$$\text{Total: } \binom{5}{3} 2! \cdot 1 \cdot 1 = 20$$

$(123)(4)(5)$ $\binom{5}{3} \frac{3!}{3}$ for the first cycle
 Cycles with 1 element always
 contribute 1 Total $\binom{5}{3} 2! = 20$

$(12)(34)(5)$ Pick 2 out of 5 $\binom{5}{2}$

Circular permutations $\frac{2!}{2}$

Pick 2 out of 3 $\binom{3}{2}$

Circular permutation $\frac{2!}{2}$

There is a new contribution. Since there are 2 cycles of the same size, we divide by $2!$ since we have double counted.

Pick (12) then $(3,4)$
or (34) then $(1,2)$

We have counted these to be different but in the end they give the same permutation. Hence

$$\text{Total } \binom{5}{2} \binom{3}{2} \frac{1}{2!} = 10 \cdot 3 \frac{1}{2} = 15$$

Remark: The reasoning in the last calculation extends as follows. If there are t cycles of the same type we divide by $t!$ since any of the $t!$ permutations on these t elements give the same result

$$(12)(3)(4)(5) : \left(\frac{5}{2}\right) \frac{2!}{2} = 10$$

TABLE

Monomial

Number

Element is
 S_5
 Action on S
 = set of edges

Element	Identity	x_1^{10}	1
(12)(2)(3)(4)(5)	$(e_2 e_5)(e_3 e_6)$	$x_1^4 x_2^3$	10
(12)(3)(4)(1)	$(e_2 e_6)(e_3 e_5)(e_4 e_7)(e_9 e_{10})$	$x_1^2 x_2^4$	15
(12)(34)(5)	$(e_1 e_5 e_2)(e_3 e_6 e_8)(e_4 e_9 e_7)$	$x_1 x_3^3$	20
(123)(4)(5)	$(e_1 e_5 e_2)(e_3 e_7 e_8 e_9 e_6)$	$x_1 x_3 x_4$	20
(123)(45)	$(e_1 e_5 e_8 e_3)(e_2 e_6)(e_4 e_9 e_{10})$	$x_2 x_4^2$	30
(1234)(5)	$(e_1 e_5 e_8 e_{10} e_4)(e_2 e_6 e_9 e_3 e_7)$	x_5^2	24
(12345)			

The cycle index: $f(x_1, x_2, x_3, x_4, x_5, x_6) =$

$$\frac{1}{120} [x_1^{10} + 10x_1^4 x_2^3 + 15x_1^2 x_2^4 + 20x_1 x_3^3 + 20x_1 x_3 x_4 \\ + 30x_2 x_4^2 + 24x_5^2]$$

To find the number of graphs:

$$f(2, 2, 2, 2, 2, 2) = 34$$

The cycle index for S_n is given in a complicated formula after a long derivation is given on page 467 of the text). We omit the details

As usual, from the cycle index we can compute the pattern inventory for graphs of order n . Here, there are only two "colors", either an edge is present or it isn't. Following the text, let 1 be assigned if the edge is not present and ε is assigned if it is present.

Ex The cycle index when $n=5$

Has just been computed. The pattern inventory is

$$f(1+\varepsilon, 1+\varepsilon^2, 1+\varepsilon^3, 1+\varepsilon^4, 1+\varepsilon^5, 1+\varepsilon^6) \\ = 1 + \varepsilon + 2\varepsilon^2 + 4\varepsilon^3 + 6\varepsilon^4 + 6\varepsilon^5 + 6\varepsilon^6 \\ + 4\varepsilon^7 + 2\varepsilon^8 + \varepsilon^9 + \varepsilon^{10})$$

So there are 6 non-equivalent graphs with 5 edges, 8 with at most 3 edges and 14 with at least 6 edges.

Problems

1. For $n=4$ find the cycle index for S_4 acting on the set \mathcal{E} of all possible edges of undirected graphs with 4 vertices. Find the number of non equivalent graphs. Construct the pattern inventory. How many graphs have 3 edges?
2. Repeat problem 1 for $n=6$.
3. Use the pattern inventory when $n=5$ to find the number of non equivalent graphs with 4 edges.