

CHAPTER 3
COMBINATORICS

MORE GRAPHS

More Graph Theory

Let $G = (V, E)$ be a graph

If $x, y \in V$ and $(x, y) \in E$, write $x \sim y$

If $x = x_1 \sim x_2 \sim \dots \sim x_n = y$, then we

say there is a walk from x to y

If $x = y$, we say the walk is closed, otherwise it is open.

If the walk has distinct vertices, except perhaps $x_1 = x_n$, then it is called a path. If $x_1 = x_n$, it is called

a cycle. A cycle with n vertices has n edges. It will be denoted by C_n



We look at several classes of graphs.

1. If $E = \emptyset$ (no edges) the graph is

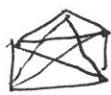
called a null graph

2. G is called complete if every

pair of vertices is joined by an edge. It is denoted by K_n (n vertices)



K_4



K_5

The degree of a vertex is the number of edges adjointed to it. Each vertex in K_n has degree $n-1$.

A connected graph is one in which for each $x, y \in V$, there is a walk from x to y .

A connected graph is called a tree if it has no cycles.

Suppose G is a tree. Take $x_1 \in V$.

Since G is connected there is $x_2 \in V$ such that $x_1 - x_2$. If $x_3 \in V$, either

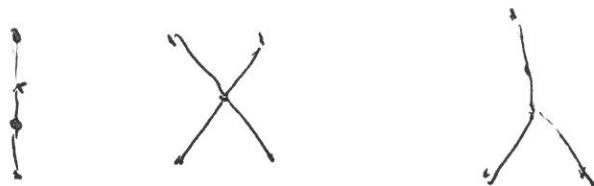
$x_3 - x_1$ or $x_3 - x_2$. Continue this way each time increasing the number of vertices by one and the number of

edges by one. In the end, there

are n vertices and $n-1$ edges in the tree. This is the smallest

number of edges in a connected graph of order n .

Ex



All are trees

G is bipartite if V can be partitioned into sets X and Y such that both X and Y are null graphs

Ex $V = \{1, 2, \dots, 10\}$ $m, n \in E$ iff

2 does not divide $m-n$

Then $X = \{1, 3, 5, 7, 9\}$ $Y = \{2, 4, 6, 8, 10\}$
shows that G is bipartite

Remark G is bipartite iff all its cycles have even length

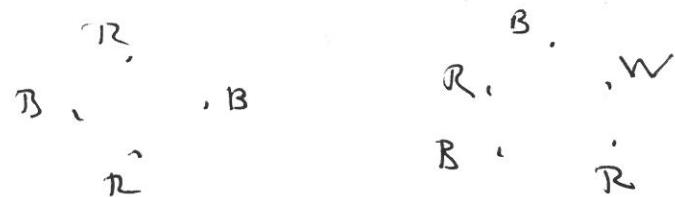
In the example: $1-4-5-8-1$

One way in the proof is easy, if G is bipartite, then its cycles are even follows as in the example

Chromatic Numbers

Let $G = (V, E)$ be a graph and S be a set with k elements, called colors. A coloring of G is an assignment of colors to each vertex such that adjacent vertices have different colors.

It is called a k -coloring.



Given G , what is the smallest k that will give a k -coloring?

It is denoted by $\chi(G)$.

Ex 1. If G is a null graph, $\chi(G) = 1$

2. If G is a complete graph K_n ,

$$\chi(G) = n - 1$$

3. $G \in C_n$, n even, $\chi(G) = 2$

4. $G \in C_n$, n odd, $\chi(G) = 3$

5. G is a tree, $\chi(G) = 2$

- Proof 1. $\chi(G) = 1$ iff G has no edges
2. If $G = K_n$, every vertex is adjacent to every other vertex, so $\chi(G) = n$
 If $G \neq K_n$, there are 2 vertices that are not adjacent so can have the same color. Hence $\chi(G) < n$
3. $G = C_n$ n even: Alternate the 2 colors
4. $G = C_n$, n-odd. Alternating leaves the final vertex connected to 2 different colors so it must have its own color
5. If G is a tree, pick a vertex and color it B. Every vertex adjacent can be colored R since there are no cycles. Then every vertex not yet colored and adjacent to the last set can be colored R. ETC.

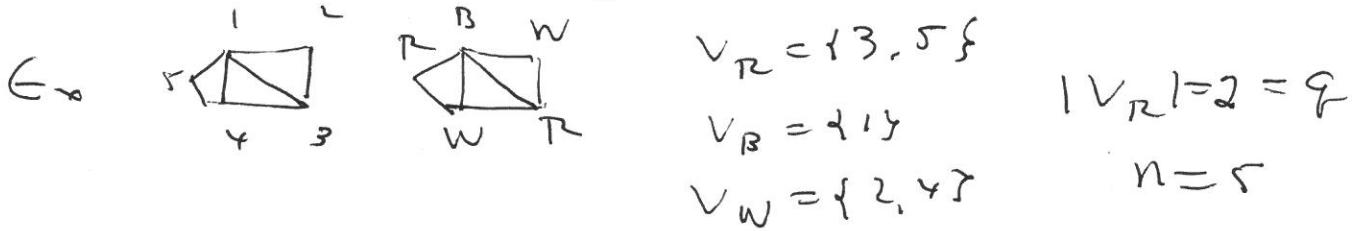
Prob Color



Let G be a graph, $S = \{1, \dots, k\}$ be a coloring. Let $V_i = \{v \in V \mid v \text{ has color } i\}$. Then V_i is a null graph since each $v \in V_i$ has color i . Hence $\chi(G) \leq k$. The set V_1, \dots, V_k is a color partition of G .

Thm. Let $G = (V, E)$, $|G| = n$. Let $\chi(G) = k$ with color partition V_1, \dots, V_k and let $|V_j| \geq |V_i| \quad j > i$. Let $|V_i| = q$. Then $n = |V| = |V_1| + \dots + |V_k| \leq k|V_i| = kq$

$$\chi(G) = k \geq \frac{n}{q}$$



$$3 = k \geq \frac{5}{2}$$

Thm. Let G have at least one edge. $\chi(G) = 2$ iff G is bipartite.

Proof. If G is bipartite, G partitions into 2 null graphs. One can have one color and the other can have a second color. $\chi(G) = 2$. Let $X = \{\alpha\}$ vertices of one color, $Y = \{\beta\}$ vertices of the other color. This shows G is bipartite.

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Thm. Suppose G is not the null graph
 $\chi(G) = 2$ iff every cycle has even length

Proof By previous results:

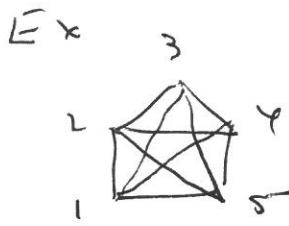
$$\begin{aligned} \text{Every cycle has even length} &\iff \\ G \text{ is bipartite} &\iff \\ \chi(G) = 2 & \end{aligned}$$

The following result describes a method to find an upper bound for $\chi(G)$; in particular $\chi(G) \leq \Delta + 1$ where Δ is the maximum of the degrees of the vertices of G .

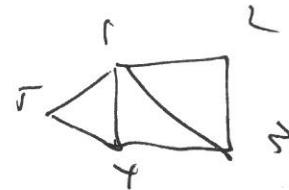
Let $V = \{v_1, \dots, v_n\}$ Assign 1 to x_1 .
For x_2 assign the smallest number not assigned to any vertex in $\{x_1\}$, so ≤ 2 .
For x_3 assign the smallest number not assigned to any of x_1, x_2 adjacent to x_1 ; ≤ 3 .
For $x_{\Delta+1}$ assign the smallest number not assigned to any x_i , x_i adjacent to $x_{\Delta+1}$
 $\leq \Delta + 1$

From then on, there are no more than Δ at the previous vertices adjacent to the rest of the vertices, so they are assigned something $\leq \Delta + 1$.

Hence $\chi(G) \leq \Delta + 1$



$$\Delta = 4$$



$$\Delta = 4$$

Vertex color

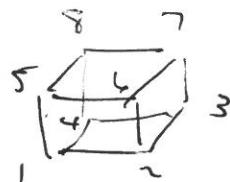
1	1
2	2
3	3
4	4
5	5

Vertex color

1	1
2	2
3	3
4	2
5	3

Problem

Apply algorithm to



what is Δ ?
what is $\chi(G)$?

The algorithm gives a bound for $\chi(G)$. It might be smaller than $\Delta + 1$. In fact, the only G where $\chi(G) = \Delta + 1$ are

K_n and C_n , n odd

Find Δ and $\chi(G)$ for and

