

REED-MULLER CODES

HADAMARD CODES

Let H be an $n \times n$ matrix whose elements are 1's and -1's and

$$HH^T = nI$$

Then $\frac{1}{n}H^T = H^{-1}$, so $H^TH = nI$ also

The dot product of any row (column) of H with itself is n and the dot product of 2 different rows is 0 (same for columns)

Multiplying any row (or column) by -1 leaves us with another such H . These matrices are called Hadamard matrices

~~and~~ we can multiply by enough -1's to obtain a Hadamard matrix whose first row (column) consists entirely of 1's. Then the dot product condition on pairs of rows (columns) gives that half the

* elements in any row (column) other than the first are 1 and half are -1. Using the dot product condition again, 2 rows have

1. 1's in $\frac{1}{4}$ of the positions

2. -1's in $\frac{1}{4}$ of the positions

3. 1 in ~~$\frac{1}{4}$~~ - first row, -1 in second in $\frac{1}{4}$ of the positions and

4. 1 in second row -1 in first in $\frac{1}{4}$ of the positions

Ex.
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

The matrix in the example was found using a general procedure

Let H be normalized Hadamard, $H \in \mathbb{R}^{n \times n}$

Then $(H^T H)$ is $2n \times 2n$ Hadamard that

is normalized (Check this)

$$\text{so let } H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, H_2 = \begin{pmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{pmatrix}$$

The matrix in the example

9) $H_3 = \begin{pmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{pmatrix}$ is 8×8 Hadamard

Note we can use this method to construct normalized Hadamard matrices of size 2^n $n=1, 2, 3, \dots$. In fact if H has size greater than 2, then it has size a multiple of 4.

Proof Let $H = (h_{ij})$ be normalized

Hadamard of size > 2 . Consider

$$\sum_{j=1}^n (h_{1j} + h_{2j})(h_{1j} + h_{3j}) = \sum_{j=1}^n h_{1j}^2 = n$$

using the dot product of rows condition. But each $(h_{1j} + h_{2j})$ is 0 or 2, same for $(h_{1j} + h_{3j})$.

So each term in the sum is 0 or 4. Hence 4 divides n .

HADAMARD CODES

Start with a $4k \times 4k$ Hadamard matrix H and we can assume it is normalized.

1. Remove the first row and first column

2. Change all -1 to 0

40 The result is a $4k-1$ by $4k-1$ matrix with $\leftarrow 2k$ position in each pair of rows that are different (see discussion above of dispersion of 1 and -1 in pairs of rows in a normalized Hadamard matrix)

Assume the rows in this matrix are the elements of a code C in $4k-1$ space. There are $4k-1$ elements in C and the designed distance is $2k$. Hence $k-1$ errors can be corrected.

Example

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix} \rightarrow J_2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

11 Reed-Muller Codes

Continue the Hadamard code example. Let K be obtained from T by interchanging all 0's and 1's. Then put a 1 in front of each row in T and a 0 in front of each row in K . Call these matrices S and T . Construct $\begin{pmatrix} S \\ T \end{pmatrix}$

This matrix has $8K-2$ rows and $4K$ columns. The designed distance is still $2k$. We can add 2 more rows, one of all 1's the other all 0's. Then

$|C| = 8K$ in $4K$ space and cor has designed distance $2k$ so can correct $k-1$ errors

This is a famous Reed-Muller code, used in sending information back from space

12 Ex Continue the last example.

Starting with J

$$\begin{array}{ccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline \text{J} & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline \end{array}$$

$$\begin{array}{ccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \end{array}$$

Fill in a 1's as a first column of J

and 0's as a first column of K

Then add a row of 1's at the top
and a row of -1's on the bottom

The result, a 16×8 matrix gives
 $|C| = 16$, length = 8, d = 4, $\tau = 1$