

M A 437

Lesson 19

Polya Theory I

Introduction

POLYTA THEORY

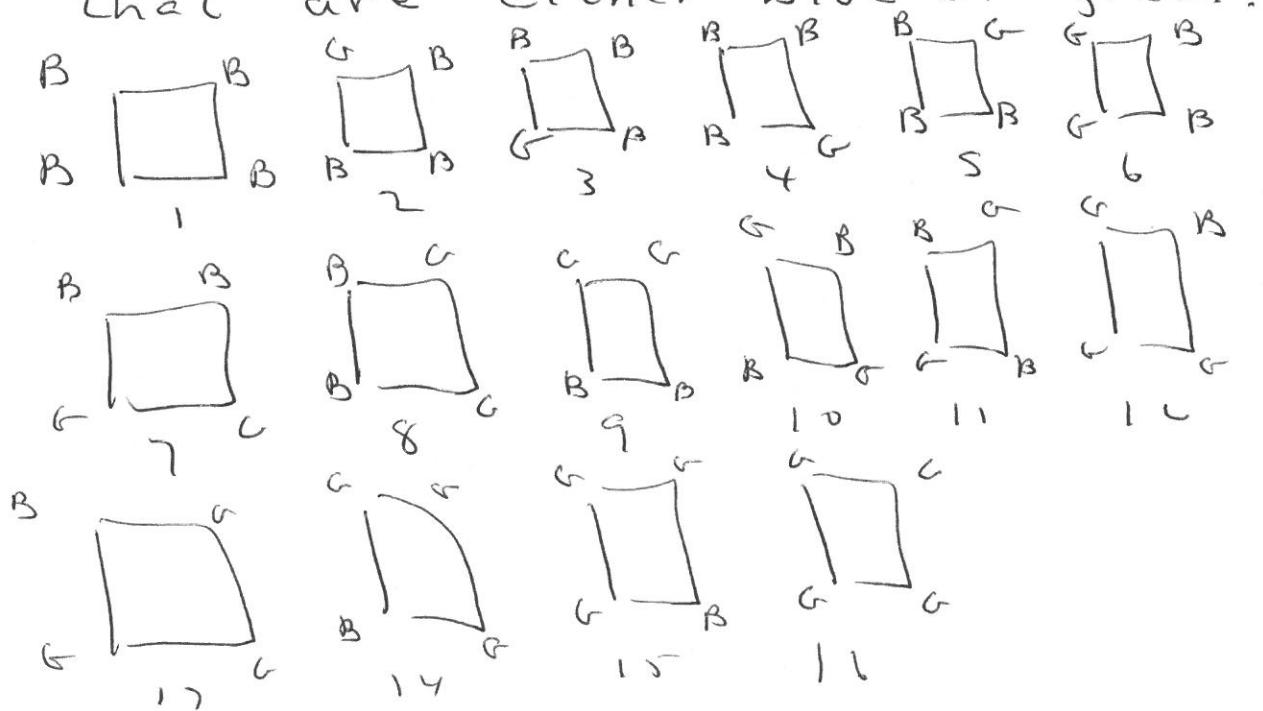
An Example

Polya theory is a counting method that uses group theory and combinatorics.

We begin with an example

Consider a necklace with 4 beads

that are either blue or green!

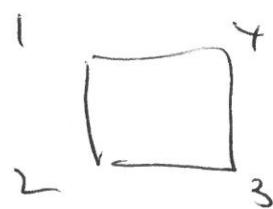


Set Σ

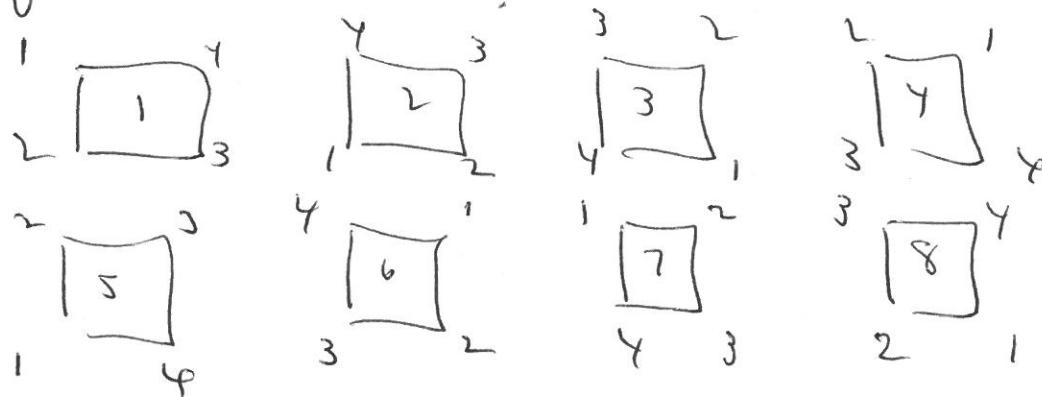
We assume the necklaces are the same if they can be moved to each other by rotations or turning them over (reflections). For example, 2, 3, 4 and 5 are the same under this definition. We need to decide what makes 2 necklaces the same, in fact, we define it. The collection of all rotations and reflections used in this problem forms a group (this will always be the case). The group is acting on a set of 16 necklaces

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We can also consider these motions to act on a square



which does not have the colors on it. The big advantage is there are fewer squares here.



Each motion is a permutation on the vertices of the square

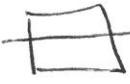
Motion

$$g_1 = \text{identity}$$

$$g_2 = 90^\circ \uparrow$$

$$g_3 = 180^\circ \uparrow$$

$$g_4 = 270^\circ \uparrow$$

$$g_5$$
 

$$g_6$$
 

$$g_7$$
 

$$g_8$$
 

Permutation

$$(1)(2)(3)(4)$$

$$(1234)$$

$$(13)(24)$$

$$(1432)$$

$$(12)(34)$$

$$(14)(32)$$

$$(17)(3)(24)$$

$$(13)(2)(4)$$

The last 4 are reflections

through the indicated line,

Def: Let G be a group and Y be a set. G acts on Y if each $g \in G$ is a permutation of Y with

product that has
 $(hg)(y) = h(g(y))$ for all $h, g \in G, y \in Y$
 $e(g) = y$

Here the group of motions acts on \underline{X} , on S and on

The set of vertices $\{1, 2, 3, 4\}$.

Def. Suppose G acts on Y
we say $x \sim y$ if there is
 $g \in G$ such that $gx = y$.

Lemma \sim is an equivalence relation
on Y .

Proof. Exercise.

Def The equivalence classes are called orbits.

Ex In \underline{X} the orbits are

$$\{\{1\}, \{2, 3, 4, 5\}, \{6, 7, 8, 9\}, \{10, 11\}\}$$
$$\{12, 13, 14, 15\}, \{16\}$$

Ex. The 90° rotation also is a permutation on X :

$$(1)(2\ 3\ 4\ 5)(6\ 7\ 8\ 9)(10\ 11)(12\ 13\ 14\ 15)(16)$$

The permutation on X is much longer than the permutation from 90° on S

Def. Suppose G acts on Y
 and $g \in G$. Let $\text{Fix}(g) = \{y \in Y \mid g(y) = y\}$

We compute $\text{Fix}(g)$ when G acts
 on X in our example

$g \in G$	$\text{Fix}(g)$	$ \text{Fix}(g) $
g_1	X	16
g_2	1, 16	2
g_3	1, 10, 11, 16	4
g_4	1, 16	2
g_5	1, 6, 8, 16	4
g_6	1, 7, 9, 14	4
g_7	1, 2, 4, 10, 11, 13, 15, 16	8
g_8	1, 3, 5, 10, 11, 12, 14, 15	8
Total	$ \text{Fix}(g) $	48

ConTrasting $\text{Fix}(g)$. we have

Def Let $y \in Y$. Define

$$S\text{Tab}(y) = \{g \in G / g(y) = y\}$$

Continue the example

$x \in X$	$S\text{Tab}(x)$	$ S\text{Tab}(x) $
1	$\underline{g_1, g_2}, G$	8
2	g_1, g_7	2
3	g_1, g_8	2
4	g_1, g_7	2
5	g_1, g_8	2
6	g_1, g_5	2
7	g_1, g_6	2
8	g_1, g_5	2
9	g_1, g_1	2
10	g_1, g_3, g_7, g_8	4
11	g_1, g_3, g_7, g_8	4
12	g_1, g_8	2
13	g_1, g_7	2
14	g_1, g_8	2
15	g_1, g_7	2
16	G	8

$$\text{Total} = 48$$

$$\sum_{g \in G} |\text{Fix}(g)| = 48 = \sum_{x \in X} |\text{stab}(x)|$$

This result always holds
Lemmas. Suppose G acts on Y . Then

$$\sum_{g \in G} |\text{Fix}(g)| = \sum_{y \in Y} |\text{Stab}(y)|$$

Proof. Let $T = \{(g, y) / g \in G, y \in Y, g(y) = y\}$

Count down the first component collecting similar g 's. The Total is $\sum_{g \in G} |\text{Fix}(g)|$. Count down the

Second component collecting similar y 's. The Total is ~~$\sum |\text{Fix}(y)|$~~

$\sum_{y \in Y} |\text{Stab}(y)|$. So the result holds

and the sums are both $|T|$.

Def Let G act on Y . For each $y \in Y$,

$$H_y = \text{Stab}(y) \quad \text{let}$$

$$\text{Orb}(y) = \{x \in Y / x = g(y) \text{ for some } g \in H_y\}$$

As we know, this action is an equivalence relation and the orbits are the equivalence classes

We are interested in the number of equivalence classes in these problems

Now suppose G acts on Y .

For each $y \in Y$, $|G| = |\text{Stab}(y)| |\text{Orb}(y)|$

Proof This follows from Lagrange's Thm. from which we know the order of $G = (\text{order of a subgroup}) (\text{number of cosets})$. Here $\text{Stab}(y)$ is a subgroup. We see that $|\text{Orb}(y)| = \text{number of cosets of } \text{Stab}(y)$.

For if $g, h \in \text{same left coset}$, Then $g = hs$, $s \in \text{Stab}(y)$. So $g(y) = hs(y) = h(y)$ so g, h ~~have~~ take y to the same element in its orbit. On the other hand, suppose $g(y) = h(y) \rightarrow y = h^{-1}g(y)$ and $h^{-1}g \in \text{Stab}(y) \rightarrow h^{-1}g = s$ for some $s \in \text{Stab}(y) \rightarrow g = hs \rightarrow g$ and h are in the same left coset. So there is a 1-1 correspondence between the left cosets of $\text{Stab}(y)$ and the elements in the orbit of y so they have the same order and the result holds by Lagrange's Thm.

Burnside's Thm.

Thm: Let G act on Y . The number of orbits in Y is

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

Proof. From the previous results

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \frac{1}{|G|} \sum_{y \in Y} |\text{Stab}(y)|$$

$$= \sum_{y \in Y} \frac{1}{|\text{Orb}(y)|}. \quad \text{Suppose the orbits are}$$

O_1, \dots, O_t . For $x, y \in O_j$,

$$|\text{Orb}(x)| = |\text{Orb}(y)| = |O_j|$$

Hence $\sum_{y \in O_j} \frac{1}{|\text{Orb}(y)|} = \frac{1}{|O_j|} + \dots + \frac{1}{|O_j|} = 1$

Hence

$$\sum_{y \in Y} \frac{1}{|\text{Orb}(y)|} = t, \quad \text{the number of}$$

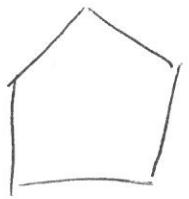
relations \Rightarrow

Ex. In our example, $\sum_{g \in G} |\text{Fix}(g)| = 48$

$$\text{So } \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \frac{1}{8} \cdot 48 = 6 =$$

number of orbits (number of different necklaces)

Notice if the necklace has 5 beads



, then there are

5 rotations and 5 reflections.

So $|G|=10$. To compute the number of orbits, we need to find $|\text{Fix}(g)|$, $g \in G$. We do not have to do this for all 10 elements in G since the reflections will all have the same $|\text{Fix}(g)|$ (Note this was not the case when $n=4$, and the non trivial rotations also have the same $|\text{Fix}(g)|$).

Consider  For $|\text{Fix}(g)|$,

There are 2 possibilities at 1, 2 at 2, 2 at 3 but then 4 and 5 are determined. So a total of 8 necklaces are fixed: $|\text{Fix}(g)|=8$. This holds for all 5 reflections, total $|\text{Fix}(g)| = 8 \cdot 5 = 40$ for the reflections.

... a non-trivial rotation, whatever color is at position 1, it will need to be at each other position for the necklace to be fixed, so 2 for each rotation and 4 rotations give 8.

For the identity, all necklaces are fixed. There are 32 of them.

$$\sum |Fix(g)| = 40 + 8 + 32 = 80$$

Number of necklaces \leq

Number of orbits =

$$\frac{1}{10} \sum_{g \in G} |Fix(g)| = \frac{80}{10} = 8$$

The problem is a little more complicated for $n=6$, but not for $n=7$.