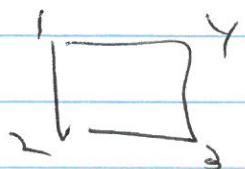


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Lesson 20

The Cycle Index and
The Pattern Inventory

Let's begin by considering again the 4 bead problem. The beads can be considered to be on the corners of a square. A group G moves the corners around.



There are 8 such moves. For example, 1 can go to one of 4 places, 2, which is connected to 1 must remain connected to 1, so 1 can go to one of two places, either side of 1. Then the rest of the corners moves are determined. Thus no more than $4 \cdot 2 = 8$ moves. There are 8 moves, 4 rotations and 4 reflections. This analogy holds for a n sided regular figure, so it has $2n$ motions. We have noted that we can list the motions as permutations of the corners!

Motion	Permutation	Monomial
120°	(1)(2)(3)(4)	$x_1 x_2 x_3 x_4 = x_1^4$
180°	(1 2 3 4)	x_4
270°	(1 3)(2 4)	$x_2 x_4 = x_2^2$
	(1 4)(2 3)	x_4
	(1 2)(3 4)	$x_2 x_4 = x_2^2$
	(1)(3)(2 4)	$x_1 x_2 x_4 = x_1^2 x_2$
	(2)(4)(1 3)	$x_1 x_2 x_3 = x_1^2 x_2$

With each cycle we assign a variable x_j if the cycle has length j . The third column in the table has the resulting monomial.

Adding these monomials and dividing by the order of the group, 6, gives an important polynomial, the cycle index of the problem.

$$f(x_1, x_2, x_3, x_4) = \frac{1}{8} (x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2 x_2)$$

We use it to count the number of $\text{Fix}(g)$ which gives us the number of orbits.

3

To see this, consider the last motion in the table. If a color is put at corner 1 and the motion is performed, then, for the picture to stay the same, the same color needs to be at corner 3. Any color can be at corner 2 and any color can be at corner 4, hence we get $2 \cdot 2 \cdot 2 = 8$ choices. For the second motion a color at corner 1 determines the colors at corners 2, 3 and then 4. So we get two choices of colors. In general, any cycle must be made up of the same colors. So in motion 3, since there are 2 cycles, there are 2 choices for cycle 1 and 2 for cycle 2, a total of 2^2 . Filling out the whole list gives

$$2^4 + 2 + 2^2 + 2 + 2^2 + 2 + 2^3 + 2^3$$

The exponents are the number of cycles, and the 2^1 's are the number of colors. We can get this by

$$f(2, 2, 2, 2) = \frac{1}{8} (2^4 + 2 \cdot 2 + 3 \cdot 2^2 + 2 \cdot 2^3)$$

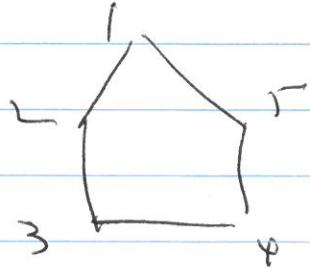
4

$$\frac{4^8}{8} = 6 \text{ orbits}$$

Notice that if we have 3 colors
we compute

$$f(3, 3, 3, 3) = \frac{1}{8} (3^4 + 2 \cdot 3 + 3 \cdot 3^2 + 2 \cdot 3^3 \\ = \frac{1}{8} (168) = 21 \text{ orbits} \\ = (\text{necklaces})$$

Ex We compute the cycle index
for a pentagon



Identify x_1^5
each rotation x_5
each reflection x_1, x_2

$$f(x_1, x_2, x_3, x_4, x_5) = \frac{1}{10} (x_1^5 + 4x_5 + 5x_1x_2^2)$$

How many necklaces are there
using beads of 2 colors?
or 3 colors?

5

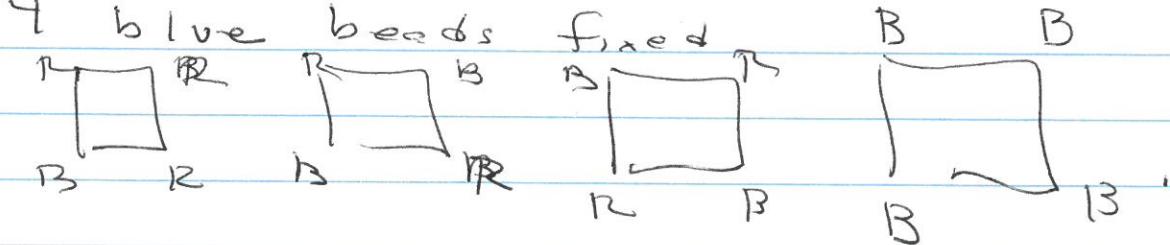
We now ask a harder problem. How many of the necklaces have exactly 2 red beads?

First consider the second permutation, 90° rotation. It's monomial is X_4 . This permutation, a cycle can have all 4 red or all 4 blue beads. Either R^4 or B^4 . Replace X_4 by $R^4 + B^4$ and we have the possible combinations, $R^4 = 4$ red or $B^4 = 4$ blue.

Now suppose we consider the 180° rotation with the permutation

$X_2 X_2$. Do as before replace X_2 by $B^2 + R^2$ to get $(R^2 + B^2)(R^2 + B^2) = R^4 + 2R^2B^2 + B^4$. This says on

necklace has all 4 red, one two have 2 R and 2 B and 1 has all



We do this for each monomial and add!

$$f(x_1, x_2, x_3, x_4) = \frac{1}{8} (X_1^4 + 2X_4 + 3X_2^2 + 2X_1^2 X_2)$$

Given

6

$$f(R+B, R^2+B^2, R^3+B^3, R^4+B^4) =$$

$$\frac{1}{8} [(R+B)^4 + 2(R^4+B^4) + 3(R^2+B^2)^2]$$

$$+ 2(R+B)(R^2+B^2)] =$$

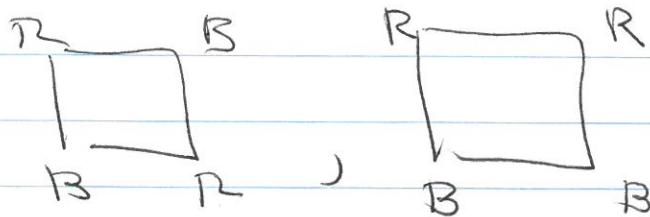
$$= \frac{1}{8} [R^4 + 4R^3B + 6R^2B^2 + 4RB^3 + B^4 - 2R^4 - 3R^2 - 6R^2B^2 - 3B^4$$

$$- 2R^4 + 4B^3B + 4R^2B^2 + 4RB^3 - 2B^4]$$

$$= \frac{1}{8} (8R^4 + 8R^3B + 16R^2B^2 + 8RB^3 + 8B^4)$$

$$= R^4 + R^3B + 2R^2B^2 + RB^3 + B^4$$

There are 2 types of beads
with 2 red and 2 blue;



$$\text{Here } f(R+B, R^2+B^2, R^3+B^3, R^4+B^4)$$

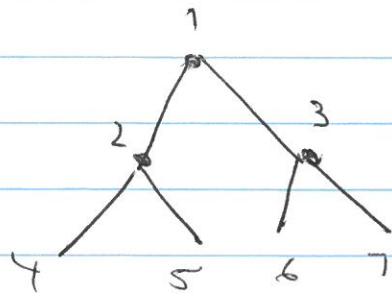
$$= R^4 + R^3B + 2R^2B^2 + RB^3 + R^4$$

is called the pattern inventory
for the problem

PROBLEMS

- 1.a Find the group G for a triangle. Write it in permutations
- b Find the cycle index
- c. How many necklaces are there using 2 colors, 3 colors
- d. Find the pattern inventory for 2 colors, 3 colors
- e. How many different necklaces with all 3 colors used
- f. How many with exactly 2 of the colors used.
- 2 Repeat this problem for a pentagon (5 beads)
- 3 Repeat this problem with a hexagon.

An ornament looks like



It can rotate 180° at
1, at 2 and at 3

We are to put lights at
2, 3, 4, 5 and 6.

Find

- The permutations of the rotations
- The order of group of rotations
- The cycle index
- The number of ornaments

using 2 colors at

2, 3, 4, 5, 6, 7

- In a 5 bead necklace with 3 colors, how many necklaces are there with only 2 of the colors used?