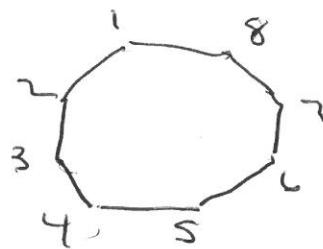


LESSON 21

POLY A THEORY PROBLEMS

We begin with another example:

Consider an 8 bead necklace
(a regular octagon)



Compute its cycle index

Motion	Permutation	Monomial
Identity	$(1)(2)(3)(4)(5)(6)(7)(8)$	x_1^8
$45^\circ \uparrow$	$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$	x_8
$90^\circ \uparrow$	$(1\ 3\ 5\ 7)(2\ 4\ 6\ 8)$	x_4^2
$135^\circ \uparrow$	$(1\ 4\ 7\ 2\ 5\ 8\ 3\ 6)$	x_8
$180^\circ \uparrow$	$(1\ 5)(2\ 6)(3\ 7)(4\ 8)$	x_2^4
$225^\circ \uparrow$	$(1\ 6\ 3\ 8\ 5\ 2\ 7\ 4)$	x_8
$270^\circ \uparrow$	$(1\ 7\ 5\ 3)(2\ 8\ 6\ 4)$	x_4^2
$315^\circ \uparrow$	$(1\ 8\ 7\ 6\ 5\ 4\ 3\ 2)$	x_1^8 x_8
	$(1\ 8)(2\ 7)(3\ 6)(4\ 5)$	x_2^4
	$(1)(5)(2\ 6)(3\ 7)(4\ 8)$	$x_1^2 x_2^3$

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There are 3 more reflections
 of each of the two types
 in the last 2 rows of the
 Table, hence we get the same
 monomial. Then the cycle
 index is

$$f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \\ \frac{1}{180} [x_1^8 + 3x_8 + 2x_4^2 + 5x_2^4 + 4x_1^2 x_2^3]$$

The number of necklaces where
 2 colors are used is
 $f(2, 2, 2, 2, 2, 2, 2)$

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Switching Functions

A switching function is a function $f: \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$

Ex. $n=2$

$$\begin{aligned}f(0,0) &= 1 \\f(1,0) &= 0 \\f(0,1) &= 1 \\f(1,1) &= 0\end{aligned}$$

They are used in the design of digital computers. The number of them gets big fast. For a given n , the number is 2^{2^n} . When $n=5$, there are more than 4 billion of them! Records are kept by defining an equivalence relation on them and keeping track of one in each equivalence class. Let X be all possible switching functions on \mathbb{Z}_2^n . Let S_n act on X by, if $\pi \in S_n$

$$\pi(f(x_1, \dots, x_n)) = f(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)})$$

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$\Leftarrow n=3 \quad \pi = (1\ 2\ 3)$

$$\pi(f(x_1, x_2, x_3)) = f(x_3, x_1, x_2)$$

Suppose $f(1, 1, 0) = 1, \quad f(0, 1, 1) = 0$

Then $\pi(f(1, 1, 0)) = f(0, 1, 1) = 0$

$\Leftarrow n=2 \quad$ Let f be the switching function $f(0, 0) = 1, \quad f(0, 1) = 0$

$$f(1, 0) = 1 \quad f(1, 1) = 0$$

Let $\pi = (1\ 2)$

$$\pi f(0, 0) = f(0, 0) = 1$$

$$\pi f(1, 0) = f(0, 1) = 0$$

$$\pi f(0, 1) = f(1, 0) = 1$$

$$\pi f(1, 1) = f(1, 1) = 0$$

So if $g(0, 0) = 1, \quad g(0, 1) = 0, \quad g(1, 0) = 1, \quad g(1, 1) = 0$

Then f is equivalent to g under this definition.

Def $f, g \in X$. $f \sim g$ if there is a $\pi \in S_n$ such that $\pi f = g$ (as in the last example)

We will use Poly's Theory to count the number of equivalence classes (orbits) in X under the group action of S_n acting on X .

Ex. Consider $n=3$. For $\pi \in S_3$, π can be displayed as

$$\pi_1 \quad (1)(2)(3)$$

$$\pi_2 \quad (12)(3)$$

$$\pi_3 \quad (13)(2)$$

$$\pi_4 \quad (23)(1)$$

$$\pi_5 \quad (123)$$

$$\pi_6 \quad (132)$$

We list how each π acts on \mathbb{Z}_2^3

Label	\mathbb{Z}_2^3	π_1	π_2	π_3	π_4	π_5	π_6
1	000	000	000	000	000	000	000
2	001	001	001	100	010	100	010
3	010	010	100	010	001	001	100
4	000	011	101	100	011	101	110
5	100	100	010	001	100	010	001
6	101	101	011	101	011	101	011
7	110	110	111	111	111	111	111
8	111	111	111	111	111	111	111

Now we list how each π acts

on rows 1-8

$$\pi_1: (1)(2)(3)(4)(5)(6)(7)(8)$$

$$\pi_2: (1)(2)(35)(46)(7)(8)$$

$$\pi_3: (1)(25)(3)(47)(6)(8)$$

$$\pi_4: (1)(23)(4)(5)(67)(8)$$

$$\pi_5: (1)(253)(467)(8)$$

$$\pi_6: (1)(235)(47)(1)(8)$$

Monomial

$$x_1^8$$

$$x_1^4 x_2^2$$

$$x_1^4 x_2^2$$

$$x_1^4 x_2^2$$

$$x_1^2 x_3^2$$

$$x_1^2 x_3^2$$

$$f(x_1, x_2, x_3) = \frac{1}{6} [x_1^8 + 3x_1^4 x_2^2 + 2x_1^2 x_3^2]$$

The colors are 0 and 1

$$f(2, 2, 2) = \frac{1}{6} (480) = 80$$

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Notice that permutations in S_3
give the same monomial if
they have the same cycle structure.
This gives us a way to save
a lot of work. If we
can count the number of permutations
with the same cycle structure,
we need only find the monomial
for one of them and then
multiply by the number with
that cycle structure. In the
last example π_2 , π_3 and π_4
have the same cycle structure,
so we multiply $x_1^2 x_2^2$, the monomial
for any of them by 3. A
similar thing is done for π_5 and π_6 .

This leads to finding a way to
count the number of cycle structures
for a particular type.

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We demonstrate with an example.

Suppose $n = 5$.

The different cycle types correspond to the partitions of n , (the ways of writing n as a sum of positive integers. For $n=5$, we get

$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$

5 stands for one 5 cycle. We have 5 choices for the first position, 4 for the second, 3 for the third, 2 for the fourth and 1 for the fifth. A total of $5!$ But these are the same

$54321, 43215, 32154, 21543, 15432$

There are 5 of these. So the number of different permutations of length 5 in this example is $\frac{5!}{5} = 4!$

$4+1$ gives permutations (abcd)

Again there are $5 \cdot 4 \cdot 3 \cdot 2$ choices for a, b, c, d. We have $5!$ ways but for each cycle there are 3 more that are the same,

9 Abcd, bcd_a, cdab, dabc. So

We divide by 4 : $\frac{5!}{4} = 30$

For $S=3+2$, There are ~~5!~~ $\frac{5 \cdot 4 \cdot 3}{3!}$

for the first cycle and $\frac{2 \cdot 1}{2!}$ for
the second, for example

(abcd)(de). Then

$$\frac{5 \cdot 4 \cdot 3}{3} \cdot \frac{2 \cdot 1}{2} = 20$$

For $S=3+2+1$, set ~~5!~~

$$\frac{5 \cdot 4 \cdot 3}{3} \cdot \frac{2}{1} \cdot \frac{1}{1} \quad \text{But the } + \text{ben}$$

But a number being repeated, like
1 here, adds a complication.

because picking a then b is

the same as picking b then a

so we need to divide by 2, actually

$$2!. \text{ So we get } \frac{5 \cdot 4 \cdot 3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1} \cdot \frac{1}{2} = 20$$

For $S=2+2+1$. Get

$\frac{5 \cdot 4}{2} \cdot \frac{3 \cdot 2}{2} \cdot \frac{1}{1}$. The pair of
2 means we divide by $2!$ too set

$$\frac{5 \cdot 4}{2} \cdot \frac{3 \cdot 2}{2} \cdot \frac{1}{2} = 15$$

10 For $S = 2 + 1 + 1 + 1$, set $\frac{5 \cdot 4}{2} + \frac{3}{1} \cdot \frac{2}{1} \cdot \frac{1}{1} \cdot \frac{1}{3!} = 1^0$

TABLE

Permutation Number

5	$\frac{5!}{5} = 4! = 24$
4+1	$\frac{5 \cdot 4 \cdot 3 \cdot 2}{4} \cdot \frac{1}{1} = 30$
3+2	$\frac{5 \cdot 4 \cdot 3}{3} \cdot \frac{2 \cdot 1}{2} = 20$
3+1+1	$\frac{5 \cdot 4 \cdot 3}{3} \cdot \frac{2}{1} \cdot \frac{1}{1} \cdot \frac{1}{2} = 20$
2+2+1	$\frac{5 \cdot 4}{2} \cdot \frac{3 \cdot 2}{2} \cdot \frac{1}{1} \cdot \frac{1}{2} = 15$
2+1+1+1	$\frac{5 \cdot 4}{2} \cdot 3 \cdot 2 \cdot 1 \cdot \frac{1}{3!} = 10$
1+1+1+1+1	1

$$\text{Total} = 24 + 30 + 20 + 20 + 15 + 10 + 1 = 120$$

Ex. When $n=7$, find the number of permutations of type $(2, 2, 2, 1)$

$$\frac{7 \cdot 6}{2} \cdot \frac{5 \cdot 4}{2} \cdot \frac{3 \cdot 2}{2} \cdot 1 \cdot \frac{1}{3!}$$

$$= 7 \cdot 3 \cdot 5 = 105$$

Ex. Compute the cycle index for S_5

Permutation	Number	Monomial
(12345)	24	x_5
$(1234)(5)$	30	$x_4 x_1$
$(123)(45)$	20	$x_3 x_2$
$(123)(4)(5)$	20	$x_3 x_1$

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(1 2)(4 5)	20	$x_3 x_1^2$
(1 2)(3 4)(5)	15	$x_2^2 x_1$
(1 2)(3)(4)(5)	10	$x_2 x_1^3$
(1)(2)(3)(4)(5)	1	x_1^5

$$f(x_1, x_2, x_3, x_4, x_5) =$$

$$\frac{1}{120} \left[24x_5 + 30x_4x_1 + 20x_3x_2 \right. \\ \left. + 20x_3x_1^2 + 15x_2^2x_1 + 10x_2x_1^3 + x_1^5 \right]$$

Problems

1. Compute the cycle index for $S_3 \times_4$ and S_6

2. A flag has 5 vertical stripes. There are 3 colors that can be used. The flag also can be turned around and used. What is the cycle index? What number of flags are possible?

HOMWORKS SET 7

I. Six beads are to make up a necklace. The necklace can be turned around and rotated.

What is the cycle index?

How many necklaces can be made using 2 colors?

II A flag has 8 vertical stripes. The flag can be turned around. If there are 3 colors to be used, find

a. The cycle index

b. The number of flags

c. How many flags are there if exactly 2 of the colors are to be used.

III Find the cycle index for S_6