

Lesson 23

Graphs

1 Review and Remarks

We considered both the necklace problem and graph problem. In the necklace problem we counted the number of necklaces with n beads of k different colors, the whole set of which we called X . We solved the problem by looking at how the dihedral group, D_n , acted on the set of uncolored beads arranged in a circle. In the graph problem we looked at how the group S_n acted on the set of all possible edges in ^a the graph. The set of all graphs was denoted by X and all edges by S .

Summarizing

2

Necklace		Graph
All necklaces	X	All graphs
Uncolored beads	S	The possible edges
The color of beads	Colors	0, 1
D_n	groups	S_n

In both cases we want the number of ~~uncolored~~ non-equivalent elements in X and use S and G to compute a cycle index and then substitute the number of colors into the cycle index.

We considered the pattern inventory in the necklace problem. We also have pattern inventory in the graph problem!

When the graphs have n vertices, how many non-equivalent ones are there with k edges;

As in the necklace problem, we use the cycle index. Instead of the different ~~color~~ colors, we use 1 (for edge not there) and ϵ (for edge there)

Ex In the graphs with $n=5$ vertices we computed the cycle index to be

$$f(x_1, \dots, x_6) = \frac{1}{120} (x_1^{10} + 10x_1^4x_2^3 + 15x_1^2x_2^4 + 20x_1x_3^3 + 20x_1x_3x_4 + 30x_2x_4^2 + 24x_5^2)$$

The number of different graphs was found:

$$f(2, 2, 2, 2, 2, 2) = 34$$

To compute the pattern inventory

$$f(1+\epsilon, 1+\epsilon^2, 1+\epsilon^3, 1+\epsilon^4, 1+\epsilon^5, 1+\epsilon^6) =$$

$$y = 1 + \varepsilon + 2\varepsilon^2 + 4\varepsilon^3 + 6\varepsilon^4 + 6\varepsilon^5 + 6\varepsilon^6 + 4\varepsilon^7 + 2\varepsilon^8 + \varepsilon^9 + \varepsilon^{10}$$

(found with computer algebra package)

There are 6 non-equivalent graphs with 5 edges, 8 with at most 3 edges and 14 with at least 6 edges

Computer Alg packages:

Python

Maple

Sage

5

G R A P H S

A graph G consists of a finite set, V , of elements called vertices and a set of pairs of elements from V , called edges, and denoted by E . So $E = \{ (a, b) / a, b \in V \}$

We will consider only undirected graph so that $(a, b) = (b, a)$

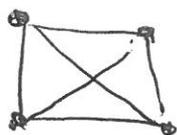
In the definition of a graph, only one edge can join a and b . There is a case when we want multiple edges joining a and b . Such graphs are called multigraphs.

A graph is called complete if

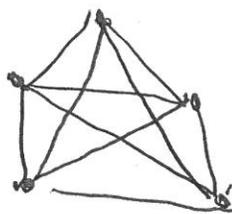
Every pair of vertices is connected by an edge. Example, $V = \{a, b, c\}$, $E = \{(a, b), (a, c), (b, c)\}$. A complete graph with n vertices is denoted by K_n . K_n has $\frac{n(n-1)}{2}$ edges

Since each of the n vertices is connected to the other $n-1$ vertices. We divide by 2 since $(a, b) = (b, a)$ for all a and b

K_4

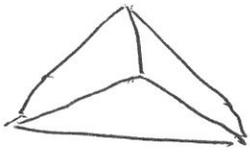


K_5



Notice that in these examples the edges might intersect at points that are not vertices. If the graph can be drawn so that this does not happen,

then the graph is called a planar graph. K_4 is actually planar



but K_n , $n > 4$, is not.

We have 2 pictures for K_4 .

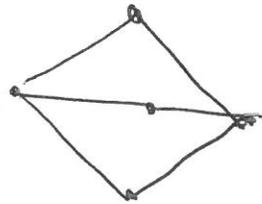
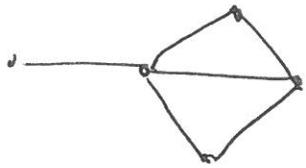
In some sense, they must be the same.

Def. Two graphs are isomorphic if there is a bijection between their vertices such that the edges in the first graph correspond to the edges in the second graph. In our K_4 graphs, any bijection of the vertices will work since all the edges appear in both graphs, but this is not necessarily true.

8
If G and G' are isomorphic
then they have the same
number of vertices and
the same number of edges.

However, the converse is false

Ex



Both graphs have 5 vertices
and 6 edges but are not
isomorphic since the first
graph has a vertex with
one edge and the second
graph does not

9
In Graph G with n vertices,
let d_1, \dots, d_n stand for the
number of edges at the
 n vertices, arranged so that
 $d_l \geq d_j$ for $l \geq j$

In the last example, the degree
sequences are

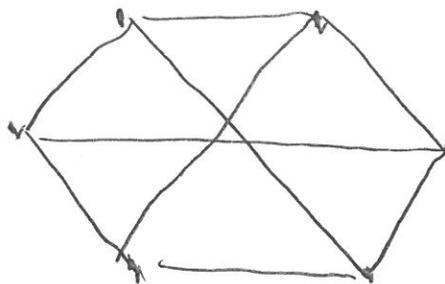
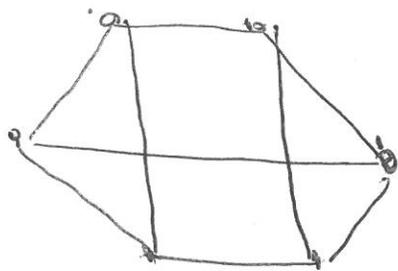
~~(4, 3, 2, 2, 1)~~ and

(3, 3, 2, 2, 2)

Notice that the sum of
the d_i in each example is
even. This is always true since
each edge adds 2 to the
total number in the sequence

It is clear that isomorphic
graphs have the same degree
sequences. The converse is false

10 Ex



Both graphs have degree seq.
(3, 3, 3, 3, 3, 3)

They can not be isomorphic
as the first graph includes
a triangle and the second does
not.

Note that the degree
sequence for K_n is

$$(n-1, n-1, \dots, n-1)$$

$\underbrace{\hspace{10em}}$
n of these

11
We have some general definitions for multigraphs (including graphs)

Let $G = (V, E)$ be a multigraph.

We consider a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{m-1}, x_m)$ and often denote it by

$$\textcircled{e} x_0 - x_1 - x_2 - \dots - x_{m-1} - x_m.$$

1. Such a sequence is called a walk

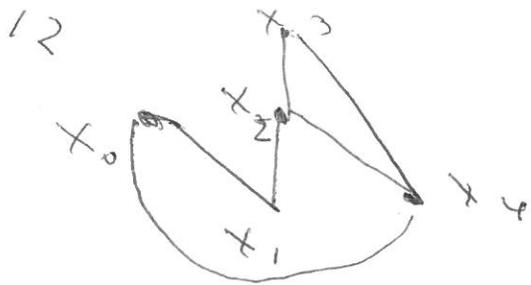
It is closed if $x_0 = x_m$

It is open if $x_0 \neq x_m$.

2. A walk with distinct edges is called a trail

3. A trail with distinct vertices is called a path

4. A closed path is called a cycle



$$x_0 - x_1 - x_2 - x_3 - x_4 - x_2 - x_1 - x_0$$

is a closed walk but not a trail

$$x_0 - x_1 - x_2 - x_3 - x_4 - x_2 - x_0$$

is a closed trail but not a path

$x_0 - x_1 - x_2 - x_3 - x_4 - x_0$ is a path which is closed so it is a cycle

$x_0 - x_1 - x_2 - x_3 - x_4$ is a path but not a cycle

For a multigraph



$x_0 - x_1 - x_2 - x_1 - x_0$ is a trail but not a path

We can always obtain a cycle from a closed walk by eliminating part of the walk: $x_0 - x_1 - x_0$