

Lesson 25

Chromatic Numbers

I. Bipartite Graphs

Def. A graph $G = (V, E)$ is called bipartite if its vertex set can be partitioned into sets X and Y such that there are no edges between vertices in X and no edges between vertices in Y .

Ex. Let $G = (V, E)$, $V = \{1, 2, \dots, 10\}$ and for $a, b \in V$, $(a, b) \in E$ if and only if $a - b$ is odd. So $(1, 2) \in E$ but $(1, 3) \notin E$.

$$\begin{aligned} \text{Let } X &= \{1, 3, 5, 7, 9\} \\ Y &= \{2, 4, 6, 8, 10\} \end{aligned}$$

These sets show that G is bipartite.

Def. A graph with no edges is called a null graph. In the example, both X and Y are null graphs.

Ex. is bipartite

is not bipartite

Theorem. $G = (V, E)$ is bipartite

if and only if each of its cycles has even length

(We are assuming that G is connected)

Proof. Suppose G is bipartite with sets X and Y as in the definition. Then any cycle must alternate between X and Y . Since the cycle must end in the set it started in, it must have an even number of edges, so it is even.

3 Suppose that each cycle has even length. Pick $x \in V$.

Let $X = \{v \in V \mid \text{shortest distance from } v \text{ to } x \text{ is even}\}$ and $Y = \{v \in V \mid \text{shortest distance from } v \text{ to } x \text{ is odd}\}$. We will show that X (and Y) are null graphs so that they make G bipartite. Suppose $a, b \in X$ and (a, b) is an edge. Let

$\alpha: x - \dots - a$ be a path to a
 $\beta: x - \dots - b$ be a path to b

There is a least common element in these paths (it might be x)

Let it be z

Consider $\alpha: x - \dots - z - t - \dots - a$
 $\beta: x - \dots - z - s - \dots - b$

Let $\alpha_1: x - \dots - z - t - \dots - a$
 $\beta_1: x - \dots - z - s - \dots - b$

Claim α_1 and β_1 have the same length

If not suppose α_1 is shorter than β_1 , then α_1, β_1 is shorter than $\alpha = \alpha_1, \beta = \beta_1$ from x to b , a contradiction

4

Since α and β are even

since a and b are in X ,

α_2 and β_2 are both even or odd.

Then $z \dots a - b - \dots - t$

$\underbrace{\quad}_{\alpha_2} \underbrace{\quad}_{1} \underbrace{\quad}_{\beta_2}$

is odd and is a cycle, a

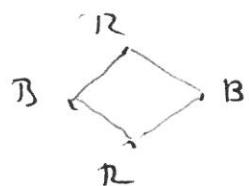
contradiction to all cycles are
assumed even. Hence $\frac{\alpha-b}{G}$ and b
does not exist and X is a
null graph. So is Y , so G is bipartite.

5

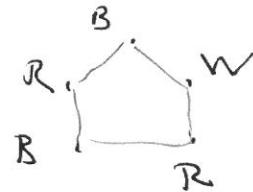
Chromatic Numbers

Let $G = (V, E)$ be a graph and S be a set with k elements, called colors. A coloring of G is an assignment of colors to each vertex such that adjacent vertices have different colors.

It is called a k -coloring.



2-coloring



3-coloring

Given G , what is the smallest k that will give a k -coloring?

It is denoted by $\chi(G)$

Ex 1. If G is a null graph, $\chi(G) = 1$

2. If G is a complete graph K_n ,

$$\chi(G) = n - 1$$

3. $G \in C_n$, n even, $\chi(G) = 2$

4. $G = C_n$, n odd, $\chi(G) = 3$

5. G is a tree, $\chi(G) = 2$

6

- Proof 1. $\chi(G) = 1$ iff G has no edges
2. If $G = K_n$, every vertex is adjacent to every other vertex, so $\chi(G) = n$
 If $G \neq K_n$, there are 2 vertices that are not adjacent so can have the same color. Hence $\chi(G) < n$
3. $G = C_n$ n even: Alternating the 2 colors
4. $G = C_n$, n -odd. Alternating leaves the final vertex connected to 2 different colors so it must therefore have its own color
5. If G is a tree, pick a vertex and color it B. Every vertex adjacent can be colored R since there are no cycles. Then every vertex not yet colored and adjacent to the last set can be colored B. Etc.

Problem. Color



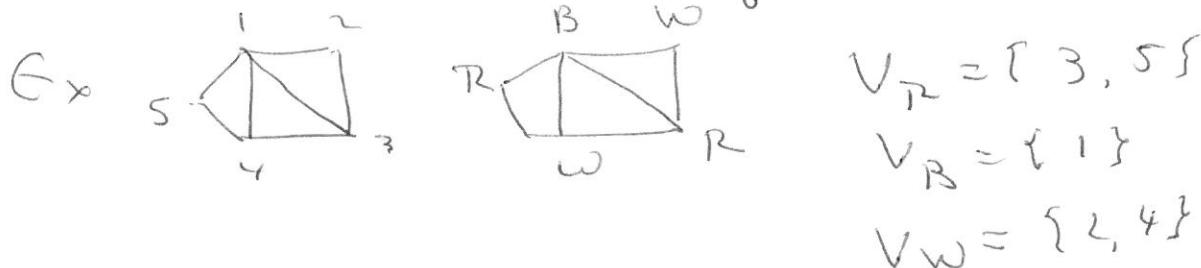
7

Let G be a graph, $S = \{1, \dots, k\}$ be a coloring of G . Let $V_i = \{v \in V / v \text{ has color } i\}$. Then V_i is a null graph. The set $\{V_1, \dots, V_k\}$ is called a color partition of G .

Theorem. Let $G = (V, E)$, $|G| = n$. Let $\chi(G) = k$ with color partition V_1, \dots, V_k . Suppose $|V_i| \geq |V_j|$ for all $i < j$ and let $g = |V_1|$. Then $\chi(G) \geq n/g$

Proof. $n = |V| = |V_1| + \dots + |V_k| \leq kg$

$$\rightarrow \chi(G) = k \geq n/g$$



$$|V_R| = 2 = g, \quad n = 5$$

$$3 = \chi(G) = k \geq \frac{n}{g} = \frac{5}{2}$$

8

Thm. Let G have at least one edge.

$$\chi(G) = 2 \text{ iff } G \text{ is bipartite}$$

Proof. If G is bipartite, G partitions into 2 null graphs, X and Y . Color the vertices in X one color and those in Y another color $\rightarrow \chi(G) = 2$.

Conversely, if $\chi(G) = 2$, let $X = \{v / v \text{ has one color}\}$
 $Y = \{v / v \text{ has the other color}\}$

X and Y are null graphs so G is bipartite.

Thm. Suppose G has at least one edge. Then $\chi(G) = 2$ iff every cycle has even length.

Proof Every cycle has even length
 iff

G is bipartite
 iff

$$\chi(G) = 2$$

The following result describes

a method to find an upper bound bound for $\chi(G)$.

In particular, $\chi(G) \leq \Delta + 1$ where Δ is the maximum of the degrees of the vertices (degree at vertex = number of edges at vertex)

Let $V = \{v_1, \dots, v_n\}$

Assign 1 to v_1 . Let $X_1 = \{v_1\}$

Assign to the smallest possible

number to v_2 considering edges to

elements in X_1 . Let $X_2 = \{v_1, v_2\}$

Note that v_2 has 1 if (v_1, v_2) is not an edge
 v_2 has 2 if (v_1, v_2) is an edge

Repeat this with v_3 and X_2

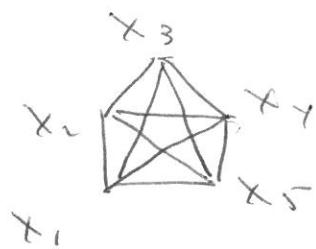
The number assigned v_3 depends on edges from v_3 to elements of X_2

v_3 gets smallest possible. So it is ≤ 3 . Repeat this. Each time

v_j and X_{j-1} , there can be no more than Δ edges from v_j to elements in X_{j-1} , so v_j gets no more than $\Delta + 1$

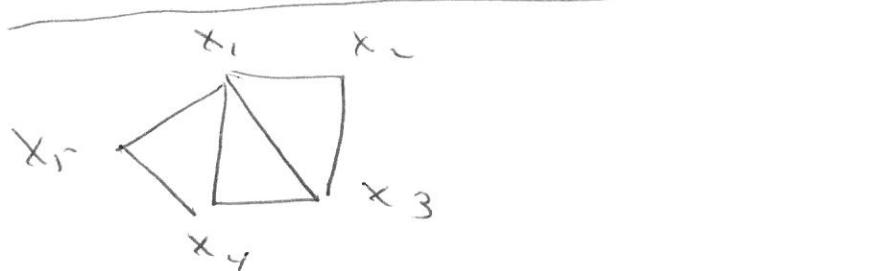
As this holds for all j , $\chi(G) \leq \Delta + 1$

Example

This is K_5

Vertex	Color	$\Delta = \gamma$
--------	-------	-------------------

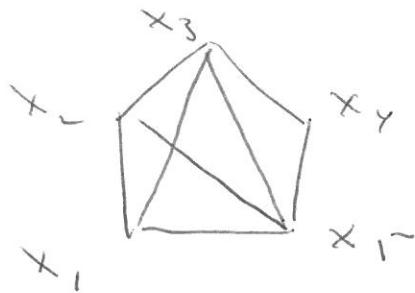
x_1	1	$\chi(\omega) = 5$
x_2	2	
x_3	3	
x_4	4	
x_5	5	



Vertex	Color
--------	-------

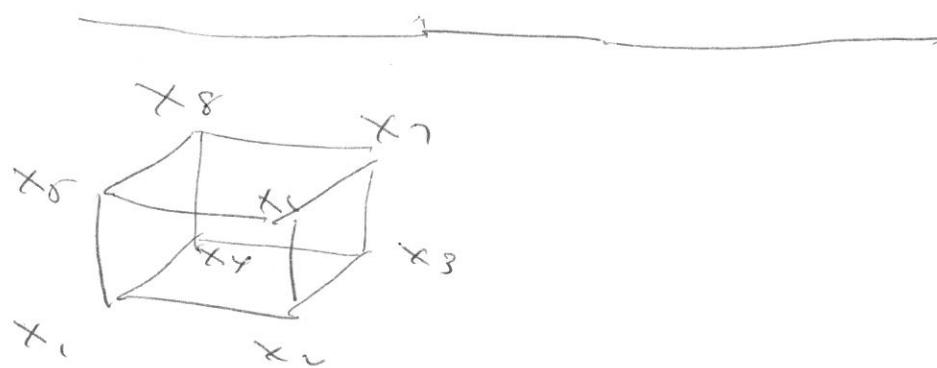
x_1	1	$\Delta = \gamma$
x_2	2	$\chi(\omega) = 3$
x_3	3	
x_4	2	
x_5	3	

11



MAKE THE TABLE

WHAT IS Δ ? $\chi(G)$?



MAKE THE TABLE

WHAT IS Δ ? $\chi(G)$?

DO YOU SEE SEVERAL OF OUR THEOREMS?

Note The algorithm gives a bound for $\chi(G)$. It may be far smaller. In fact, $\chi(G) = \Delta + 1$ if and only if $G = K_n$ or $G = C_n$ n odd