

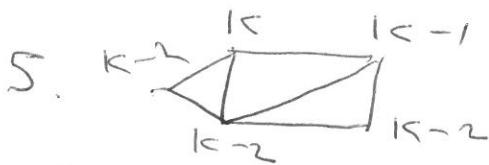
# Chromatic Polynomials II

In the last class we introduced the chromatic polynomial which is used to count the number of colorings that a graph has using  $k$  colors. The notation was  $P_G(k)$ . We saw several examples

1. If  $G$  is a tree of order  $n$ , then  $P_G(k) = k(k-1)^{n-1}$ .
2. If  $G$  is a null graph of order  $n$ , then  $P_G(k) = k^n$
3. If  $G = C_4$ , then  $P_G(k) = k(k-1)^3 + k(k-1)(k-2)^2$

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4. If  $G$  is a complete graph,  $K_n$ , then  $P_G(k) = k(k-1)\dots(k-n+1)$



Picking the order of the vertices carefully helps here

$$P_G(k) = k(k-1)(k-2)^3$$

We saw an algorithm which reduced any graph to null graphs. A. null graph with  $n$  vertices has  $k^n$  colorings. A collection of null graphs has the number of colorings found by adding the colorings of each, giving a polynomial in  $k$ . This algorithm is hard to do by hand for almost all cases but a computer will do it. It is useful in

- This gets complicated fast.  
 If computer algebra package would have the algorithm available what the algorithm shows us is we get a polynomial after some steps. Also
1. The degree of  $P_G(k)$  = number of vertices
  2. The constant in  $P_G(k)$  is 0 since we have added null graph  $P_{G(k)}$ 's and they are not constant
  3. In connected graphs, one of the  $k$  appears as a first degree term. The coefficient is the number of null graphs with one vertex and the sign is obtained from  $|V|-1 =$ 
    - +1 if  $|V|-1$  is even
    - 1 if  $|V|-1$  is odd
  4. If  $G$  has just 2 connected components  $G_1$  and  $G_2 \rightarrow P_G = P_{G_1} P_{G_2}$ . This is shown by doing the reduction algorithm first on  $G_1$ , then  $G_2$

4 This requires some details, but not many. In this case  $P_{G(K)}$  will have a  $K^L$  term but no  $K$  term (check what happens when you multiply 2 polynomials, each with a  $K$  term but no constant term) More generally,  $G$  has 5 connected components iff  $K^5$  is the smallest degree in  $P_G(K)$

5. It is, the coefficient of  $K^{n-1}$  is  $-m$  where ~~minus~~<sup>is -m where</sup> the number of edges in the graph. The algorithm and induction shows this (Induction on the number of edges).

If  $G$  has  $m$  edges

$G_1$  has  $m-1$  edges

$$\rightarrow P_{G_1}(K) = K^n - (m-1)K^{n-1} + \dots$$

$$P_{\text{fused}} = K^n - \dots$$

$$\rightarrow P_G = P_{G_1} - P_{\text{fused}} = K^n - m K^{m-1} + \dots$$

and the result appears

The most important property of  $P_G(k)$  is that for any  $k$ ,  $P_G(k)$  is the number of colorings of  $G$ .

Ex How many coloring of a square's vertices are there using

1.  $k = 2$  colors:

$$P_G(2) = 2 + 2 \cdot 1 \cdot 0 = 2$$

2.  $k = 3$  colors

$$\begin{aligned} P_G(3) &= 3 \cdot 2^3 + 3 \cdot 2 \cdot 1 \\ &= 24 + 6 = 30 \end{aligned}$$

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When working with the algorithm,  
think like this

1. An edge is removed from  $G$

$$\square \rightarrow \square \text{ to get } G,$$

2.  $G$ , can have opposite vertices

the same or different

$$\square = \begin{matrix} A & A \\ \square & \square \end{matrix}$$

3. The second,  $\square$ , is in 1-1  
correspondence with  $G$

4. The first has the vertices  
fused,  $\Delta$ , so has one less  
vertex, and is in 1-1 correspondence  
with the fused graph

5. Combining 3 and 4

$$\square = \square + \begin{matrix} A & B \\ \square & \square \end{matrix} = \square + \Delta$$

$$\rightarrow G = \square - \Delta$$

6. Continue with, a.  $\square$  and  
b.  $\Delta$ , in the same way

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Recall

$$\xrightarrow{\quad} P_G(k) = k^2 - k$$

$$\triangleright P_G(k) = (k^3 - k^2) - (k^2 - k)$$

$$\Delta \Rightarrow \triangleright = \begin{array}{c} A \\ B \end{array} + \begin{array}{c} A \\ A \end{array}$$

$$\Rightarrow \Delta = \triangleright - \begin{array}{c} \\ \end{array}$$

$$\begin{aligned} P_G(k) &= [(k^3 - k^2) - (k^2 - k)] - (k^2 - k) \\ &= k^3 - 3k^2 + 2k \end{aligned}$$

Note

$$P_G(2) = 8 - 12 + 4 = 0$$

$G$  is not 2-colorable

## 8 Computing chromatic polynomials

$$\text{Ex 1. } \square : \square = \begin{matrix} A & B & A \\ | & | & | \\ \square & \square & \square \end{matrix} + \begin{matrix} A \\ | \\ \square \end{matrix}$$

$$= \square + \triangle$$

$$\square = \square - \triangle$$

$$P_G(k) = P_{G_1}(k) - P_F =$$

$$k(k-1)^3 - k(k-1)(k-2) =$$

$$k^4 - 4k^3 + 6k^2 - 3k$$

$$\text{Ex 2. } \begin{matrix} \square \\ G \end{matrix} : \begin{matrix} \square \\ G_1 \end{matrix} = \begin{matrix} A & B \\ | & | \\ \square & \square \end{matrix} + \begin{matrix} A & A \\ | & | \\ \square & \square \end{matrix}$$

$$= \begin{matrix} \square \\ G \end{matrix} + \begin{matrix} \square \\ F \end{matrix}$$

$$\begin{matrix} \square \\ G \end{matrix} = \begin{matrix} \square \\ G_1 \end{matrix} - \begin{matrix} \square \\ F \end{matrix}$$

$$P_G(k) = k(k-1)^4 - (k^4 - 4k^3 + 6k^2 - 3k)$$

$$\text{Ex 3. } \begin{matrix} \square \\ G \end{matrix} : \begin{matrix} \square \\ G_1 \end{matrix} = \begin{matrix} A & A \\ | & | \\ \square & \square \end{matrix} + \begin{matrix} A \\ | \\ \square \end{matrix}$$

$$= \begin{matrix} \square \\ G \end{matrix} + \begin{matrix} \square \\ F \end{matrix}$$

$$\begin{matrix} \square \\ G \end{matrix} = \begin{matrix} \square \\ G_1 \end{matrix} - \begin{matrix} \square \\ F \end{matrix}$$

$$P_G(k) = (k^4 - 4k^3 + 6k^2 - 3k) - k(k-1)^2$$

Ex 4

$$\begin{aligned}
 G &= G_1 + A \cdot B + A \cdot A \\
 &= G_1 + F \\
 G &= G_1 - F
 \end{aligned}$$

$G_1$  is  $K_4$  with extra vertex and 2 edges

$$P_{G_1}(k) = k(k-1)(k-2)(k-3)(k-2)$$

$$F: P_F(k) = k(k-1)(k-2)(k-3)$$

$$P_G(k) = P_{G_1}(k) - P_F(k) = k(k-1)(k-2)(k-3)(k-2-1)$$

Ex 5  $G$  is  $n$  cycle CLA(m)

$$P_G(k) = (k-1)^n + (-1)^n (k-1)$$

Induct on  $n$

$$\begin{aligned}
 n=3: \quad \textcircled{1} : \textcircled{2} &= \textcircled{3}^A + \textcircled{3}^B = \textcircled{1} + \textcircled{2}
 \end{aligned}$$

$$\textcircled{1} = \textcircled{2} - \textcircled{3}$$

$$P_G(k) = k(k-1)^2 - (k-1)^3 = (k-1)^3 + (-1)^3 (k-1)$$

Assume for  $G = C_{n-1}$ ,  $P_G = (k-1)^{n-1} + (-1)^{n-1} (k-1)$

$$\text{For } C_n: \quad \textcircled{1} : \textcircled{2} = \textcircled{3}^A + \textcircled{3}^B = \textcircled{1} + \textcircled{2}$$

$$\Rightarrow \textcircled{1} = \textcircled{2} - \textcircled{3} \quad \text{tree } n-1 \text{ cycle}$$

$$P_n(k) = \left| k \right| \left( (-1)^{n-1} - \left[ (-1)^{n-1} + (-1)^{n-1} \right] \right)$$

$$= \left( (-1)^n + (-1)^n \right) \rightarrow$$

Notice that in the examples the sign alternates in the monomials. This is true because each time we eliminate a vertex it is from the fused part and the part is subtracted. So when we get to the particular null graph, if there are  $r$  vertices then we made  $n-r$  fused steps to get there and there are  $n-r$  sign changes. The monomial we get is  $k^r$ . If there is another null graph with  $r$  vertices the exact same process says the sign in front of  $k^r$  is the same. In  $k^{r-1}$  there has been one more step and one sign change. Hence the signs alternate and the different null graphs of the same size do not cancel out, they add together for the coefficient of  $k^r$ .

## PROBLEMS

1. Why are the following NOT chromatic polynomials?

a.  $P_G(k) = k^3 - 5k^2 + k$

b.  $P_G(k) = k^3 - k^2 + 2k + 2$

c.  $P_G(k) = k^3 - 2k - 4k$

d.  $P_G(k) = k^3 - 2k$

2. Find  $P_G(k)$  for



using one step in the algorithm.

3. Find  $P_G(k)$  for



Find  $P_G(2)$  and  $R_G(3)$  for  
both 2 and 3