

LESSON 28

Block Designs

A magazine wants to compare 7 different cars by assessing the opinions of 7 different drivers. One could have all 7 drivers test all 7 cars, but more efficiency would be desirable. How can a testing schedule be set up? The method we discuss is called a balanced incomplete block design, or block design for short. Here are the principles.

Let  $S = \{q_1, \dots, q_v\}$  be a set of  $v$  objects. Let  $B_1, \dots, B_b$  be subsets of  $S$ , called blocks. The restraints are

1. Each object appears in the same number of blocks

2. Each block contains the same number of objects

3. Each possible pair of objects appears together in the same number of blocks.

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The block design is described using parameters

$\{N, b, r, k, \lambda\}$  where

$N$  = number of objects

$b$  = number of blocks

$r$  = each object is in  $r$  blocks

$k$  = each block contains  $r$  objects

$\lambda$  = number of blocks each pair appears in together

In addition, we assume

$k < N$  and  $\lambda > 0$

Both these conditions make sense

$k \leq N$  and if  $k=N$  then every block contains every object.

If  $\lambda=0$ , the ~~the~~ objects are not being compared against each other

Ex

In the car problem, let  $\{1, \dots, 7\}$  be the cars (objects) and each driver will be a block. The cars the driver tests will be the block for him

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A case that would work is

$$B_1 = \{1, 2, 4\}$$

$$B_2 = \{2, 3, 5\}$$

$$B_3 = \{3, 4, 6\}$$

$$B_4 = \{4, 5, 7\}$$

$$B_5 = \{5, 6, 1\}$$

$$B_6 = \{6, 7, 2\}$$

$$B_7 = \{7, 1, 3\}$$

Then  $n=7$ ,  $b=7$ ,  $r=3$ ,  $k=3$

and  $\lambda=1$ . This example has not been done randomly, there is a technique that we will see. This is the point of what we will do, construct block designs using various methods.

There are some relations between the parameters

Thm In a block design

$$nr = bk \text{ and } (n-1)\lambda = r(k-1)$$

Proof To show  $nr = bk$ , consider

$T = \{a, B\}$  object  $a$  is in block  $B\}$

Count  $|T|$  in two different ways

Each of the  $n$  objects occurs in  $r$  blocks, hence

$$|T| = nr$$

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Each of the  $b$  blocks contains  $k$  objects, hence  
 $|T| = bk$

Therefore  $Nr = bk$

To show  $(N-1)\lambda = r(k-1)$ , consider an object, say  $a$ . Let

$$U = \{(x, B) / x \text{ and } a \text{ are in } B\}$$

Count  $|U|$  in two ways.

There are  $N-1$  objects (beside  $a$ )

that appear in  $\lambda$  blocks with  $a$ ,

$$|U| = (N-1)\lambda$$

There are  $r$  blocks that

contain  $a$ , along with  $k-1$  objects,

$$|U| = r(k-1)$$

Therefore  $(N-1)\lambda = r(k-1)$

### Incidence Matrices

Let  $A = (a_{ij})$  where

$$a_{ij} = 1 \text{ if } a_i \in B_j$$

$$a_{ij} = 0 \text{ if } a_i \notin B_j$$

So column  $i$  stands for  ~~$B_i$~~   $B_i$

row  $j$  stands for  ~~$a_j$~~   $a_j$

$A$  is called the incidence matrix for the design

Ex In the car example,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The sum of each column is  $k$   
and each row is  $r$

We can use incidence matrices  
to create block designs and  
also to prove further results  
about them. Remember that  
multiplying row  $i$  by row  $j$  is the  
dot product of these rows. So  
multiplying row  $i$  of  $A$  by  
column  $j$  of  $A^T$  is also the  
dot product of row  $i$  and row  $j$ .  
The dot product of row  $i$  with  
itself is  $r$  and with row  $j$  is  
 $\lambda$ . Hence

$$AA^T = \begin{pmatrix} r & \lambda - \lambda \\ \lambda & r - \lambda \\ \lambda & \lambda - \lambda \\ \lambda & \lambda - r \end{pmatrix} = (r-\lambda)I + \lambda J$$

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where  $J$  is the matrix with every element = 1

Lemma Let  $B$  be the  $n \times n$  matrix

$$B = (r - \lambda)I + \lambda J \text{ Then} \\ \det B = (r - \lambda)^{n-1} (r + (n-1)\lambda)$$

Proof  $B = \begin{pmatrix} r & \lambda & \dots & \lambda \\ \vdots & \ddots & \ddots & \vdots \\ \lambda & \lambda & \dots & r \end{pmatrix}$

Subtract first col of  $B$  from each other column to get

$$B_1 = \begin{pmatrix} r & \lambda-r & \dots & \lambda-r \\ \lambda & r-\lambda & \dots & 0 \\ \lambda & 0 & r-\lambda & \dots & 0 \\ \vdots & 0 & 0 & \dots & r-\lambda \end{pmatrix}$$

Add the first row of  $B_1$  to each other row of  $B_1$  to get

$$B_2 \begin{pmatrix} r+(n-1)\lambda & 0 & \dots & 0 \\ \lambda & r-\lambda & \dots & 0 \\ 0 & \vdots & \ddots & 0 \\ \lambda & 0 & \dots & r-\lambda \end{pmatrix}$$

$$\det B = \det B_1 = \det B_2 \\ = [r + (n-1)\lambda] (r - \lambda)^{n-1}.$$

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Theorem In any block design,  $n \leq b$  and  $k \leq r$ .

Proof let  $A$  be the incidence matrix  
Since we assume  $k < n$  always,  
 $\lambda < r$  by the first Theorem.

Then  $\det AA^t \neq 0$  by the Lemma  
Always  $\text{rank } A \geq \text{rank } AA^t = n$

$A$  is  $n$  by  $b$ , it follows that  
 $n \leq b$  since the ranks of any  
matrix is ~~less~~ than the number  
of rows (columns)

Def A block design is called symmetric  
if  $n = b$ . Our example is  
symmetric. By the first  
Theorem  $r = k$  for symmetric  
block designs

Thm. In a  $(n, n, r, r, \lambda)$  (symmetric)  
block design, every pair of  
blocks will contain exactly  
 $\lambda$  objects in common.

The proof is in the text  
on page 33. It uses incidence  
matrices.

# Construction

## Hadamard Matrices

There are various methods to construct Block Designs.

We will look at several.

The first uses Hadamard matrices, a topic we have looked at already. Recall that

$$\mathbf{H}$$

Def A Hadamard matrix is an  $n \times n$  matrix whose entries are  $\pm 1$  and

$$\mathbf{H} \mathbf{H}^t = n \mathbf{I}$$

The equation says that

$$\mathbf{H}^{-1} = \frac{1}{n} \mathbf{H}^t$$

and that  $\mathbf{H}^t \mathbf{H} = n \mathbf{H}^{-1} \mathbf{H} = \mathbf{H}(n \mathbf{H}^{-1}) = \mathbf{H} \mathbf{H}^t$ , so  $\mathbf{H}$  and  $\mathbf{H}^t$  commute,  $\mathbf{H} \mathbf{H}^t = \mathbf{H}^t \mathbf{H}$

The dot product of distinct rows (columns) is 0 in a Hadamard matrix. Changing the sign in any row (column) of  $\mathbf{H}$  gives another Hadamard matrix.

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Hence, starting with  $H$ , we can change the sign of any row (column) getting a Hadamard matrix with 1's in both the first row and first column.

Such a Hadamard matrix is called normalized. We will always start with normalized Hadamard matrices. Then each other row (column) contains half the elements 1 and the other half -1 since such a dot product is 0. Thus  $n$  is even. In fact, if  $n > 2$ ,  $n$  must be a multiple of 4. For let

$$H = (h_{ij})$$

Then

$$\sum (h_{i,j} + h_{2,j})(h_{i,j} + h_{3,j}) = \sum h_{i,j} = n$$

Now each term in the sum is 0 or 4 (check that each  $h$  is 1 or -1 to see this). Hence 4 divides  $n$ .

There is an algorithm for constructing these matrices

$$\text{let } H_1 = (1) \quad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{pmatrix}$$

$$H_4 = \begin{pmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{pmatrix}. \text{ Continue this.}$$

Once we have  $H_n$  let

$$H_{2n} = \begin{pmatrix} H_n & H_n \\ H_n & -H_n \end{pmatrix}$$

This gives Hadamard matrices of order  $2, 4, 8, 16, \dots 2^n$ .

There exist Hadamard matrices of other orders but not obtained this way

### Block Designs

We can construct the incidence matrix for a Block Design from any normalized Hadamard matrix

Start with  $H$  of order  $4t \times 8$

Delete the first row and column

Change all  $-1$  to  $0$ . This is an incidence matrix for a

$(4t-1, 4t-1, 2t-1, 2t-1, t-1)$  block.

Proof. After doing the algorithm, we have a matrix  $A$ . Each row and column have  $4t-1$  elements,  $2t-1$  are 1. This gives the first four parameters in the Block Design. In any pair of rows in  $A$ , there are  $2t$  positions where the entries differ,  $t$  positions where both contain +1,  $t$  where both contain -1,  $t$  position where both are -1 and  $t-1$  where both are 1. The dot product is  $\lambda = t-1$ . Hence

$$AA^T = (r-\lambda)I + \lambda J,$$

all  $4t-1$  matrices

$A$  is the incidence matrix for the block design we claimed.

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$$\text{Ex } H_8 = \begin{pmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{pmatrix}$$

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$H_8 = \left( \begin{array}{cccc|cccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array} \right)$$

Cross out 1st row, 1st col then  
change  $-1 \rightarrow 0$

$$\left( \begin{array}{cccc|cccc|c} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 6 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{array} \right)$$

This is  $(?, ?, 3, 3, 1)$

$$= (4t, 4t-1, 2t-1, 2t-1, t-1)$$

design

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Now to set up the block design, the object  $N_5$  goes into block  $B_1$  if there is a 1 in the  $(c, j)$  position. So object  $N_1$  is in blocks  $B_2, B_4$  and  $B_6$ . Likewise, block  $B_1$  contains objects  $N_2, N_4$  and  $N_5$  because that is where 1 appears in the first column

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## PROBLEM

Suppose a magazine wants to obtain a comparison of 15 cars using 15 drivers.

Construct a block design to do this. What are the parameters,  $(n, b, r, k, \lambda)$