

Lesson 2

Permutations

We discuss an example that shows the difference between two important concepts.

Ex. A city council has 10 members. They are setting up a committee for a certain project.

- (a) There will be four members on the committee, a president, a vice president, a secretary and a treasurer. In how many ways can the committee be formed. In order, there are 10 choices for the president, then 9 for the V.P., 8 for the Secretary and 7 for the Treasurer. The total is

$$10 \cdot 9 \cdot 8 \cdot 7 = 5040 = \frac{10!}{6!}$$

2. (b) Suppose that instead, we just want a committee of 4. How many ways can they be chosen? Temporarily denote the answer by A. A is the number of ways of picking 4 out of 10 where order does not matter. We use the answer to part (a).

4 out of 10 can be picked in A ways. Then they can be assigned the roles in $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ ways. The multiplication principle gives $A \cdot 4!$ ways. But from part (a) there are $P(10, 4) = \frac{10!}{6!}$ ways.

Therefore $A \cdot 4! = \frac{10!}{6!}$ or

$$A = \frac{10!}{4! 6!}$$

Let's assign some names to these processes and some notation

- (a) This is called the number of permutations of 4 things from 10 and denoted by

$$P(10, 4) = \frac{10!}{(10-4)!} = \frac{10!}{6!}$$

- (b) This is called the number of combinations of 4 from 10 and denoted by $\binom{10}{4} = \frac{10!}{4! 6!}$

In both cases we are picking 4 out of 10. In the first case, order matters. In the second, it does not, only the people selected. This is the distinguishing feature. In a given problem, we need to decide: Does order matter?

4. More generally, picking r out
of n is

(a) $P(n, r) = \frac{n!}{(n-r)!}$ if order matters

or d

b $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ if order does not matter

Ex From a deck of 52 cards

(a) 5 can be laid out in a row
in $P(52, 5)$ ways

(b) dealt as a hand in
 $\binom{52}{5}$ ways

In a, no order matters

In b, order does not matter.

5

Order does not matter, a hand is
a hand no matter how it came about.

$$\text{Ans } \binom{52}{5} = \frac{52!}{5! \cdot 47!}$$

Again (a) is a permutation problem
(b) is a combination problem

A problem can often be done
in two ways. We saw that in
solving the second committee problem.
Ex. From 8 types of fruit, we will
pick 3, one of each type, and then
put the 3 in order on a table.

(a) Pick the 3 where order matters
setting $P(8, 3) = \frac{8!}{5!}$
or

b) Pick 3 pieces in $\binom{8}{3}$ ways
 and the order them in $3!$ ways
 setting

$$\binom{8}{3} 3! = \frac{8!}{3! 5!} 3! = \frac{8!}{5!} = P(8, 3)$$

In permutation problem, often we use all the elements. In other words, given n elements (all different) how many ways can we list them? Order matters.

$$\text{Ans } P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Recall that $0!$ is defined to be 1.
 Ex. How many 6 letter "words" can be formed from a b c d e f.

$$\text{Ans} = P(6, 6) = 6!$$

How many 5 letter words can be formed from these 6 letters

$$\text{Ans} = P(6, 5) = \frac{6!}{(6-5)!} = 6!$$

Ex In a 6 by 6 grid, 15 numbers are to be assigned to 15 spots in the grid. How many ways can this happen? 1 can be assigned to 36 possible spots, then 2 to 35 spots, etc i.e $36 \cdot 35 \cdots [(36-15)=21]$

$$= P(36, 15) = \frac{36!}{21!}$$

Ex How can we order the 26 letters in the alphabet so that no 2 vowels are together

The consonants take up 21 places
 The 5 vowels can be in any of 22 spots (Before the first consonant, after the last consonant, or in the 20 spots between consonants. So pick 5 of the 22 spots for the vowels: $P(22, 5)$
 Order the consonants: $21!$
 Ans = $(21!) P(22, 5)$

Ex. How many 7 digit numbers are there such that 5 and 6 do not appear next to each other. We do not have 0 in the problem.

This is an example that uses the subtraction principle

There are $P(9,7)$ ways of choosing

and ordering 7 numbers out of 9

These orderings make up the universal set U . We count

the number of ways 5 and 6

can be together. These would have

5 followed by 6, with

5 in positions 1, ..., 6

or 6 followed by 5, 6 in

positions 1, ..., 6.

Then the other

7 numbers are ordered in the

remaining 5 positions in $P(7,5)$ ways

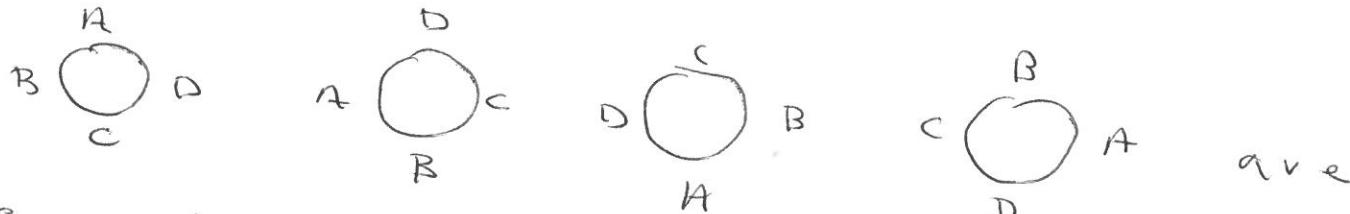
so

Ans $2 \cdot 6 \cdot P(7,5)$

For 6 first position the first of 5 or
5 or 6 first 6 is in

In the permutation problems considered so far, the objects have been lined up in a row. Suppose that they are around a circle (a so called circular permutation). How many ways can n objects be placed around a circle? There are $n! = P(n, n)$ ways of arranging them in a row. For the circle, the objects can be placed around the circle and rotating them $\frac{360}{n}$ degrees gives another arrangement that we consider to be the same. All that matters is what object is on either side of an object. There are n rotations that give the same arrangement. Thus there are $\frac{n!}{n} = (n-1)!$ circular arrangements.

Ex



Considered to be the same.

So each linear arrangement of n objects has n circular arrangements. They are the same. The number of circular arrangements is

$$\frac{P(n, n)}{n} = \frac{n!}{n} = (n-1)!$$

More generally, if n objects are to be put in r positions around a circle, the number of ways is

$$\frac{P(n, r)}{r} = \frac{n!}{r(n-r)!}$$

Ex 15 people are to be seated around a table with 10 chairs. How many circular arrangements are there?

$$\frac{P(15, 10)}{10} = \frac{15!}{(5!) 10}$$

Ex. Ten people are to be seated around a table. How many arrangements are there?

$$\frac{P(10, 10)}{10} = 9!$$

Another way to think about circular permutations is to seat someone in a particular seat and count how to seat the remaining people. There are $(n-1)!$ ways to seat the remaining people, so that is the answer. Note

$$\frac{P(n, n)}{n} = (n-1)!$$

Ex. 10 people, 2 of which do not want to sit next to each other are to be seated around a table. How many ways can this be done?

1st solution: The total number of arrangements is $\frac{P(10, 10)}{10} = 9!$. If the two sit next to each other, seating the rest can be done in $8!$ ways. There are 2 ways of seating the two people together. So using the subtraction principle:

$$9! - 2 \cdot 8! \text{ ways}$$

12

2nd solution: Seat one of the two
at the head of the table

There are 8 possible people on the left and
then 7 on the right. Then for the other
7 seats, anything is possible = $7!$

$$\text{Ans } 8 \cdot 7 \cdot 7! = 9! - 2 \cdot 8!$$

Problems Page 61

3, 7, 8, 9, 10, 11