

Lesson 3

Combinations

1.

Combinations

Recall: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ $\binom{n}{0} = 1$

Ex How many } set combinations can be

made from {a, b, c, d, e}

{a b c} {a b d} {a b e} {a c d} {a c e} {a d e}

{b c d} {b c e} {c d e} {b d e}

$$\binom{5}{3} = \frac{5!}{3!2!} = 10$$

Ex. There are 25 seats in a classroom and 15 students enrolled

(a) How many different sets of 12 students could attend? $\binom{15}{12}$

(b) The 12 students could be arranged in the room in how many ways?

The first student has choice of 25 seats
 The second " " " 24 seats
 The third " " " 23 seats

This is a permutation problem:
 $25 \cdot 24 \cdot \dots \cdot 14 = P(25, 12) = \frac{25!}{13!}$

(c) PUT (a) and (b) into one problem!
 How many ways could the 15 students have 12 chosen and then arranged:

$$\binom{15}{12} P(25, 12)$$

2. Ex a. How many 8 letter "words" are there if letters can be repeated

$$26^8$$

b. How many having 3 vowels?

We can pick the 3 positions for the vowels in $\binom{8}{3}$ ways

The number of choices for vowels in these 3 positions 5^3

The number of choices for constants in the 5 positions: 21^5

$$\text{Ans } \binom{8}{3} 5^3 21^5$$

c. How many having at least 3 but no more than 5 vowels

$$\binom{8}{3} 5^3 21^5 + \binom{8}{4} 5^4 21^4 + \binom{8}{5} 5^5 21^3$$

Theorem $\binom{n}{r} = \binom{n}{n-r}$

Proof This can be done using the formula for $\binom{n}{r}$. We do it

combinatorically: Let U be a set with n elements. For each set S with r elements, there is its complement, S^c with $n-r$ elements

The map $\pi: S \rightarrow S^c$ of each set to its complement is clearly a bijection. So the number of sets

3 with r elements \Rightarrow the number of sets with $n-r$ elements : $\binom{n}{r} = \binom{n}{n-r}$

Ex. A room has 2 rows of seats, 8 seats in a row, 13 students attend class

a. How many ways can the students be seated if

a. 6 people are in row 1 and we only care which people they are

Ans $\binom{13}{6}$

b. We also care which seat they are in $\binom{13}{6} P(8,6) P(8,7)$

c. Answer question (a) if any number of students can be in row 1

$$\binom{13}{8} + \binom{13}{7} + \binom{13}{6} + \binom{13}{5}$$

d. Answer question (b) if any number can be in row 1

$$\begin{aligned} & \binom{13}{8} P(8,8) P(8,5) + \binom{13}{7} P(8,7) P(8,6) \\ & + \binom{13}{6} P(8,6) P(8,7) + \binom{13}{5} P(8,5) P(8,8) \\ & = 2 \left[\binom{13}{8} P(8,8) P(8,5) + \binom{13}{7} P(8,7) P(8,6) \right] \end{aligned}$$

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Thm Pascal
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

The left side is the number of subsets with k elements

They come in 2 disjoint Types:

Let a be an element in the universal set U . The number of

k element subsets without a is $\binom{n-1}{k}$

The number with a is $\binom{n-1}{k-1}$

i.e. without a we must pick k out of $n-1$. with a we must pick $k-1$ out of $n-1$. So

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Thm.
$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

Proof There are 2^n subsets of U

There are $\binom{n}{j}$ with j elements

Hence the left hand side counts

The number with $0, 1, 2, \dots, n$ elements

The Total = Total of subsets = 2^n

Problems P. 61

10, 12, 13, 14, 15

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Ex. A team has 20 players and 9 positions to fill

How many ways can this be done if

(a) It does not matter what position a player is in

(b) It does matter what position a player is in

(a) Order does not matter: $\binom{20}{9}$

(b) Order does matter $P(20, 9)$

6 Solutions P. 61

7.
$$\frac{4! \cdot 8!}{4}$$

8.
$$\frac{6! \cdot 6!}{6}$$

9.
$$13 \cdot 12 \cdot 12!$$

$$13 \cdot 13!$$

10. a.
$$\binom{12}{2} \binom{10}{3} + \binom{12}{3} \binom{10}{2} + \binom{12}{4} \binom{10}{1} + \binom{12}{5}$$

b. The answer to a is the number in the universal set

b. Complement, they are both on committee

b.
$$\binom{11}{1} \binom{9}{2} + \binom{11}{2} \binom{9}{1} + \binom{11}{3} \binom{9}{0}$$

Ans = a - b

