

Lesson 4

Permutations of Multisets

## MULTISETS: PERMUTATIONS

A multiset is a generalization of a set in that elements can be repeated.

Ex { Tennessee }

If  $a_1, \dots, a_k$  are elements in a set repeated  $n_1, \dots, n_k$  times, we write  $\{n_1 a_1, \dots, n_k a_k\}$ .

In our example,

$\{1T, 3e, 2n, 2s\}$

If an element is repeated in unlimited numbers, we use  $\infty$  as a repeated unlimited, so a

Th. If  $S$  is a multiset and each element has infinite repetitions and there are elements of  $k$  types each with infinite repetition number, then the number of permutations of length  $r$  is  $k \cdot k = k^r$ .

Proof There are  $k$  choices for the first position,  $k$  for the second position, ...  $k$  for the  $r$  position. And  $k \cdot k = k^r$

Then  $S$  is a multiset of  $k$  different types with  $a_1$  repeated  $n_1$  times,  $\dots$ ,  $a_k$  repeated  $n_k$  times and  $n = n_1 + \dots + n_k$ . The number of permutations of  $S$  is  $\frac{n!}{n_1! \cdots n_k!}$  (if  $n$  is the size of  $S$ )

Proof  $a_1$  goes  $n_1$  places,  $\binom{n}{n_1}$  choices

Now  $a_2$  goes  $n_2$  places,  $\binom{n-n_1}{n_2}$

Now  $a_3$  goes  $n_3$  places  $\binom{n-n_1-n_2}{n_3}$

$\vdots$   
 $a_{1k}$  goes  $n_k$  places  $\binom{n-n_1-\dots-n_{k-1}}{n_k}$

The number of permutations is the product:

$$\frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-\dots-n_{k-1})!}{n_k!(n-n_1-\dots-n_k)!}$$

Each denominator has a term which is the same as the next numerator, so they cancel, to get

$$\frac{n!}{n_1! \cdots n_k!} \quad (\text{Note } (n-n_1-\dots-n_k)! = 0! = 1)$$

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Ex what is the number of permutations  
of Tennessee?

$$\frac{8!}{1!, 3!, 2!, 2!}$$

Suppose the multiset has elements  
 $a_1, a_2$  repeated  $n_1$  and  $n_2$  times  
Let  $n = n_1 + n_2$ . The number of  
permutation of the elements in the  
multiset is  $\frac{n!}{n_1! n_2!} = \frac{n!}{(n_1)!(n-n_1)!} \cdot \binom{n}{n_1}$

There is another type of problem  
that leads to the same answer

Suppose we have  $n$  distinguishable  
objects and  $k$  boxes which hold  
 $n_1, \dots, n_k$  objects and

$$n = n_1 + \dots + n_k$$

In how many ways can the elements  
be put in the boxes; i.e. how  
many different ways can objects  
be put in boxes to give different  
arrangements.

4.  $n_1$  elements go into box 1 and can be chosen in  $\binom{n}{n_1}$  ways

$n_2$  elements go into box 2 and can be chosen in  $\binom{n-n_1}{n_2}$  ways

$n_3$  elements go into box 3 and can be chosen in  $\binom{n-n_1-n_2}{n_3}$  ways

This is repeated until the  $k$ th box where the final  $n_k$  go into the last box in  $\binom{n-n_1-\dots-n_{k-1}}{n_k}$  ways

Note that  $n-n_1-\dots-n_{k-1} = n_k$  so the last term is 1

The answer is gotten by multiplying all these together and cancel as we did in the last problem to get  $\frac{n!}{n_1! \cdot n_k!}$

Ex. 15 different pieces of candy are put in 3 boxes which hold 4, 5 and 6 pieces. How many ways can this be done?  $\frac{15!}{4! 5! 6!}$

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This problem can be viewed as a multiset problem. The objects are arranged in a row. The elements in the multiset are the elements in the boxes, each labeled by the box it came from. So there are  $n_1$  elements labeled  $q_1$ ,  $n_2$  elements labeled  $q_2$ , ..  $n_k$  elements labeled  $q_k$ . How many permutations of length  $n$  are there? This is the multiset problem with the same answer as twice before.

In the candies problem, the multiset would be

$$S = \{q, q_1, q_1, q_1, q_2, q_2, q_2, q_3, q_3, q_3, q_3, q_3\}$$

How many ways can we assign these to

1 2 3 . . . 15

the pieces of candy? Here we are assigning positions in the boxes to the candy instead of the other way around but get the same result.

A variation occurs if all the boxes hold the same number of elements and the boxes are indistinguishable i.e no box  $1, \dots, k$ . Then the answer needs to take in account that we can permute the boxes in  $k!$  ways and get the same answer. Hence the number of ways then is

$$\frac{1}{k!} \frac{n!}{n_1! \dots n_k!} \quad (n_1 = n_2 = \dots = n_k)$$

Suppose in the candy problem all boxes hold 5 elements and the boxes are indistinguishable. Then the number of ways of do assigning can be

$$\frac{1}{3!} \frac{15!}{5!5!5!}$$

## Ex Non-Attacking Rooks

In chess, rooks can move left or right, up or down as many places as desired, but no other moves.

Rooks are attacking if they can move once to get to each other.

i.e., they are in the same row or column. We search for the number of ways to put rooks on the board so that they can not attack each other. There are various variations.

Suppose there are 8 rooks on an  $8 \times 8$  board. How many ways can they be situated. In any row and any column there can not be more than 1. Since there are 8 rooks on an  $8 \times 8$  board, each row and column must have exactly one. We will assign positions going down the board. The first rook has 8 possible positions. The second now has 7 (in the second row). The third has 6

$8$  (in the third row) etc :

$$\text{Ans } 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$$

Suppose the  $8$  rooks are all colored differently. Now after we assign positions in  $8!$  ways

we assign colors. The first rook has  $8$  possible colors, the second has  $7$  ... Total  $8!$  So the answer now is  $8! 8!$ .

Suppose there are one red, three blue and four yellow rooks. This is a multi-set problem where we are assigning the 3 colors to 8 positions

$$\frac{8!}{1! 3! 4!}$$

Combining this with the first step to set

$$\frac{8!}{1! 3! 4!} 8!$$

This generalizes to any combination of colors and in fact to  $n$  rooks on an  $n \times n$  board

$$\text{giving } n! \frac{n!}{n_1! \cdot n_k!}$$

where there are  $n_1$  rooks colored 1  
 $n_2$  rooks colored 2  
etc

Moving on, the number of 9 permutations of  $\{3a, 2b, 4c\}$  is

$$\frac{9!}{3!2!4!} \text{ as our standard problem}$$

What about 8 permutation?

Break this into cases

$$a. S = \{2a, 2b, 4c\} \quad \frac{8!}{2!2!4!} = 420$$

$$b. S = \{3a, b, 4c\} \quad \frac{8!}{3!1!4!} = 280$$

$$c. S = \{3a, 2b, 3c\} \quad \frac{8!}{3!2!3!} = 560$$

$$\text{Ans} = 420 + 280 + 560$$

Problems P. 62

17, 18, 19, 20, 21, 23, 24

Page 62 #14. Classroom has 2 rows of 8 seats each. 14 students where S always sit in front, T always sit in back. How many possible seating arrangement?

We do this by focusing on the S who can seat in either row.

4 in back, 1 in front + etc

$$(\frac{5}{4})P(8,8) P(8,6) + (\frac{5}{3}) P(8,7) P(8,7) + (\frac{5}{2}) P(8,6) P(8,8)$$

#15 15 M 20 W

a. 15 couples  $\binom{20}{15} P(15,15)$

(pick 15 out of 20W, then assign W to M  
 $\Rightarrow 15 \cdot 14 \cdot 13 \dots = P(15,15)$ )

b.  $\binom{20}{10} \binom{15}{10} P(10,10)$

(same idea as a)

#16 10M 12W committee of 5, must have at least 2W

$$a. (\frac{12}{2})(\frac{10}{3}) + (\frac{12}{3})(\frac{10}{2}) + (\frac{12}{4})(\frac{10}{1}) + (\frac{12}{5})(\frac{10}{0})$$

b. Do this by finding the complement, committees where they serve together:  $(\frac{12}{1})(\frac{9}{2}) + (\frac{12}{2})(\frac{9}{1}) + (\frac{12}{3})(\frac{9}{0})$

i.e. for  $(\frac{12}{1})(\frac{9}{2})$ , The one W and M are on the committee. Thus picking one more W and 2 more men out of 11W and 9M + more of the same