

M A 416

LESSON 10

1. Today we look at 2 algorithms
that generate all subsets of a set

$$S = \{x_{n-1}, x_{n-2}, \dots, x_0\}$$
$$|S| = n$$

We represent subsets of S by
strings of 0's and 1's (binary
strings)

Ex $S = \{x_3, x_2, x_1, x_0\}$

$$A = \{x_3, x_1\} \quad 0110$$

$$B = \{x_2, x_1, x_0\} \quad 0111$$

So each x_j sets the $j+1$ position
from the right and is 1 if x_j
is in A and 0 otherwise

A listing of the subsets of $\{x_2, x_1, x_0\}$

\emptyset	000	0
$\{x_0\}$	001	1
$\{x_1\}$	010	2
$\{x_2, x_0\}$	011	3
$\{x_2\}$	100	4
$\{x_2, x_1\}$	101	5
$\{x_2, x_0, x_1\}$	110	6
$\{x_2, x_1, x_0\}$	111	7

The last column ^{consists} one of the integers whose binary representations have coefficients that are in the second column.

$$\text{Ex } 5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \quad 101$$

The integers from 0 to $2^n - 1$ appear in the final column.

$$\text{Ex Let } n = 5 \quad S = \{x_4, x_3, x_2, x_1, x_0\}$$

$$|S| = 2^5$$

$$19 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$

10011 is the listing

of coefficients.

The subset corresponding to 19 : 10011 is $\{x_4, x_1, x_0\}$

Every one of the 2^n subsets corresponds to an integer between 0 and $2^n - 1$.

We order the listing of subsets using the ordering of the integers:

$$n = 4$$

0	$0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$	\emptyset
1	$0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$	$\{x_0\}$
2	$0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$	$\{x_1\}$
3	0 0 1 1	$\{x_1, x_0\}$
4	0 1 0 0	$\{x_2\}$
5	0 1 0 1	$\{x_2, x_0\}$
6	0 1 1 0	$\{x_2, x_1\}$
7	0 1 1 1	$\{x_2, x_1, x_0\}$
8	1 0 0 0	$\{x_3\}$
9	1 0 0 1	$\{x_3, x_0\}$
10	1 0 1 0	$\{x_3, x_1\}$
11	1 0 1 1	$\{x_3, x_1, x_0\}$
12	1 1 0 0	$\{x_3, x_2\}$
13	1 1 0 1	$\{x_3, x_2, x_0\}$
14	1 1 1 0	$\{x_3, x_2, x_1\}$
15	1 1 1 1	$\{x_3, x_2, x_1, x_0\}$

We can also generate the second column by adding 0001 to a term mod 2 to get the next term

What position is $\{x_3, x_2, x_0\}$ in?

$$\{x_3, x_2, x_0\} = 1101 : 2^3 + 2^2 + 2^0 = 13$$

The thirteenth position

Algorithm for generating subsets
which have order corresponding
to order of integers:

$$\text{Let } Q_{n-1} \dots Q_0 = 0 \dots 0$$

while $Q_{n-1} \dots Q_0 \neq 1 \dots 1$ do

1) Find the smallest integer j

between $n-1$ and 0 such that

$$Q_j = 0.$$

2. Replace Q_j with 1 and all

Q_{j-1}, \dots, Q_0 with 0

This is equivalent of adding 1 mod 2

$$\text{Ex } A = (x_3, x_2, x_0)$$

$$1011$$

+ 1 mod 2 gives

$$1100 \rightarrow$$

$$\text{Set } \{x_3, x_2\} = B$$

B follows A in the subset listing

\emptyset	$x_3 \ x_2 \ x_1 \ x_0$
$\{x_0\}$	0 0 0 0
$\{x_1\}$	0 0 0 1
$\{x_1, x_0\}$	0 0 1 0
$\{x_2\}$	0 0 1 1
$\{x_2, x_0\}$	0 1 0 0
$\{x_2, x_1\}$	0 1 0 1
$\{x_2, x_1, x_0\}$	0 1 1 0
$\{x_3\}$	0 1 1 1
$\{x_3, x_2\}$	1 0 0 0
$\{x_3, x_1\}$	1 0 0 1
$\{x_3, x_1, x_0\}$	1 0 1 0
$\{x_3, x_2\}$	1 0 1 1
$\{x_3, x_2, x_0\}$	1 1 0 0
$\{x_3, x_2, x_1\}$	1 1 0 1
$\{x_3, x_2, x_1, x_0\}$	1 1 1 0
$\{x_3, x_2, x_1, x_0\}$	1 1 1 1

Several remarks

1. The second 8 sets are the same as the first 8 sets except 0 in the first position becomes 1.
2. We can get an element by adding 1 mod 2 to the preceding one.

Ex Find the next subset after

$$\{x_2, x_1, x_0\} \rightarrow 0111 \rightarrow 1000 \rightarrow \{x_3\}$$

If we look back at the table notice that the subsets can change considerably in one step. For instance $\{x_3\}$ follows $\{x_2, x_1, x_0\}$

We next consider a listing method that has the property that ~~we~~ most exactly one element changes (either is added or removed) from one step to the next. It also studies a geometric problem, which we begin with.

An n cube has 2^n corners whose coordinates are the 2^n n -tuples of 0's and 1's. There is an edge joining corners if and only if the coordinates of the corners differ in exactly one place.

A one cube is a line segment,

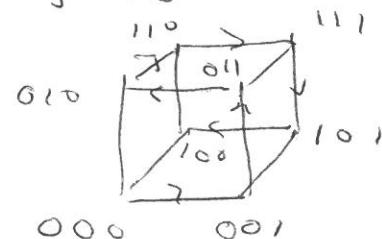
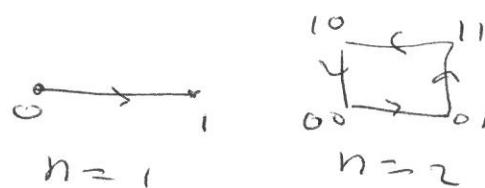
A two cube is a square and

A three cube is a regular cube.

Can we order the corners such that successive corners differ by one coordinate; i.e., a line joins them, and each corner appears exactly once. This corresponds to walking around the lines in the cube with each corner appearing once. Such a walk is called a Gray code of order n .

If it is possible to make one more step from the final corner to the beginning corner, then the Grey code is called cyclic.

Ex



To go from $n=2$ to $n=3$ construct a cyclic Gray code for $n=2$ and a second one for $n=2$ behind the first. From 010 go to 110 and go around the second code in reverse order. One obtains a Cyclic Gray code. It is called reflected since the second $n=2$ code goes in the reverse order.

~~Atmos~~ If we list the corners of a reflected Gray code and draw a line through the middle of the listing, the top and bottom (2 sides of the line) are a reflection of each other except that the first position in the top is 0 and in the bottom is 1.

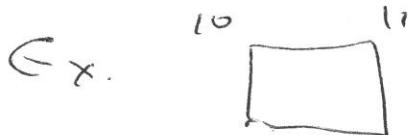
To construct the n cube,

assume inductively that one

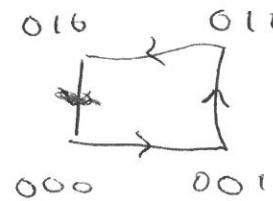
$n-1$ cube has been constructed.

List the $n-1$ tuples in order and put a 0 in front. Then list

the $n-1$ tuples in the reverse order and put a 1 in front

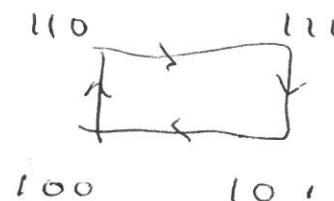


First copy



List 0 0
 0 1
 1 1
 1 0

Second copy



List in order

000
001

011

010

110

111

101

100

Reflection Line

Note we went from copy 1 to copy 2

10 When we inserted 1's in front instead of 0's in the first position

To go to $n=4$, take the $n=3$ list, put 0 in front for the first part, put 1 in front for the second part, listing the first part in forward order and the second part in reverse order we get

0 0 0 0

Notice the last

0 0 0 1

4-tuple is 1000

0 0 1 1

so there is a

0 0 1 0

line joining it and

0 1 1 1

0000 so the Gray

0 1 0 1

code is cyclic

0 1 0 0

reflecting line

1 1 0 0

1 1 0 1

1 1 1 1

1 1 1 0

1 0 1 0

1 0 1 1

1 0 0 1

1 0 0 0

An algorithm for generating these Gray codes is

Begin with n tuple 00...0

Until we get 10...0, do

the following. For sequence $q_{n-1} \dots q_0$

Compute $\sigma(q_{n-1} \dots q_0) = q_{n-1} + \dots + q_0$

If this is even, change q_0
(from 0 to 1 or 1 to 0)

If this is odd, find the first
one from the right and change
the next number to its right
(from 0 to 1 or 1 to 0)

Ex If, 0101101. $\sigma(0101101)$ is
even, so we get 0101100

If 0101100, $\sigma(0101100)$ is
odd, get 0100100

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The reflected ^{Gray} codes have the property
that only one digit changes in each
step

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11, 12, 13, 15, 16, 17

19, 20, 21, 23, 24