

M A 416

Lesson 11

# BINOMIAL COEFFICIENTS

Result 1.  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  for positive integers  $0 < k < n$

Also  $\binom{n}{k} = 0$  if  $k > n$  and

$$\binom{n}{0} = 1 = \binom{n}{n}$$

We showed identities

$$2. \binom{n}{k} = \binom{n}{n-k} \text{ and } 3. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

in chapter 2.

Also we showed Pascal's formula

$$3. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

In each case we gave a combinatorial proof. They can also be shown by algebra,

using the definition.

Pascal's Triangle

$$(0)$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

etc

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# PASCALS TRIANGLE

$$\begin{array}{ccccccc}
 & & \binom{0}{0} & & & & \\
 & \binom{1}{0} & & \binom{1}{1} & & & \\
 \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 \binom{3}{0} & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
 \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\
 & & \vdots & & & & \\
 & \dots & \binom{n-1}{k-1} & \binom{n-1}{k} & \dots & & \\
 & \dots & \binom{n}{k} & \dots & & &
 \end{array}$$

Consider  $\binom{n}{k}$ . The 2 elements diagonally above  $\binom{n}{k}$  are  $\binom{n-1}{k-1}$  and  $\binom{n-1}{k}$   
 By Pascal's Identity

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Hence the elements in the Triangle allow

easy computation.

The fifth row is given  
as

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

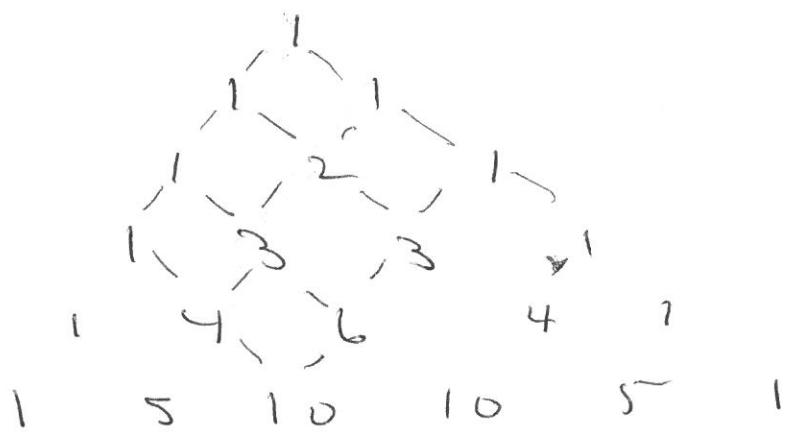
So the 6th row is

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

$$\text{and } \binom{7}{3} = \binom{6}{2} + \binom{6}{3} = 15 + 20 = 35$$

Let  $p(n, k)$  be the number  
of diagonal paths slanting down  
from  $\binom{n}{0}$  to  $\binom{n}{k}$ . The number  
is  $\binom{n}{k}$ . i.e  $p(n, k) = \binom{n}{k}$

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$$P(8,3) = 10 = \binom{5}{3}$$

Theorem  $P(n, k) = \binom{n}{k}$

Proof Clearly  $P(n, 0) = 1 = P(n, n)$  where you can only go down the outside diagonal. Hence  $P(n, 0) = \binom{n}{0}$  and

$P(n, n) = \binom{n}{n}$ . Use induction and

assume the result for  $n-1$  and show  $P(n, k) = \binom{n}{k}$ . Hence

$$P(n-1, k-1) = \binom{n-1}{k-1}, \quad P(n-1, k) = \binom{n-1}{k}$$

There is only one path from  $\binom{n-1}{k-1}$  to  $\binom{n}{k}$  and one from  $\binom{n-1}{k}$  to  $\binom{n}{k}$ .

$$\text{Hence } P(n, k) = P(n-1, k-1) + P(n-1, k) = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

# BINOMIAL THEOREM

Theorem: Let  $n$  be a positive integer.

$$\text{Then } (x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

$$\text{Ex } (x+y)^3 = \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y^1 + \binom{3}{2} x^1 y^2 + \binom{3}{3} x^0 y^3$$

Proof.  $(x+y)^n = (x+y)(x+y) \dots (x+y)$

\* For  $x^k y^{n-k}$ , we need all ways

to pick  $k$  elements  $x$  and  $n-k$  elements  $y$   
 $\hookrightarrow$  to get all  $x^k y^{n-k}$ . Thus we  
 are choosing all combinations of  $k$   $x$ 's  
 and  $n-k$   $y$ 's. That is  $\binom{n}{k}$  and the  
 result holds

$$(x+y)^5 = x^5 + \binom{5}{1} x^4 y + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 \\ + \binom{5}{4} x^1 y^4 + \binom{5}{5} x^0 y^5$$

\* the coefficient of  $x^k y^{n-k}$

$$6 \text{ Theorem } (1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

Proof Let  $y=1$  in the binomial theorem.

$$\text{Theorem } k \binom{n}{k} = n \binom{n-1}{k-1}$$

$$\text{Proof } k \binom{n}{k} = k \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

$$\text{while } n \binom{n-1}{k-1} = n \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

$$\text{Theorem } \binom{n}{0} + \dots + \binom{n}{n} = 2^n$$

Proof. This was shown in chapter 2

Here, set  $x=1=y$  in the binomial Theorem

$$2^n = (1+1)^n = \sum_{k=0}^n \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k}$$

$$\begin{aligned} \text{Theorem. } \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} &= 2^{n-1} \\ \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} &= 2^{n-1} \end{aligned}$$

Proof Set  $x=1$   $y=-1$  Then

$$(1) - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots = (1-1)^n = 0$$

$$\text{Thus } \binom{n}{0} + \binom{n}{2} + \dots = \binom{n}{1} + \binom{n}{3} + \dots$$

$\Leftarrow n=1$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5}$$

This last result says that for each  $n$ , the number of subsets with an even (odd) number of elements is  $2^{n-1}$

A combinatorial proof is instructive

Let  $S = \{x_1, \dots, x_n\}$

To construct subsets:

If  $A$  is a subset

$x_1$  has 2 choices, in  $A$  or not

$x_2$  has 2 choices, in  $A$  or not

$x_{n-1}$  has 2 choices, in  $A$  or not

Finally at  $x_n$ . ~~If~~ The choice for

$x_n$  is already determined. If

$|A|$  is even and there are an even number of  $x_i, x_{n-1}$  in  $A$ , then  $x_n$  is not in  $A$ . If there are an odd number in  $A$ , then  $x_n$  is in  $A$ . Similar reasoning holds if  $|A|$  is odd. Hence there are  $2 \cdots 2^{(n-1)}$  choices =  $2^{n-1}$

Computing binomials by calculus

Start with

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

Differentiate

$$\Rightarrow n(1+x)^{n-1} = \sum_{k=1}^n k \binom{n}{k} x^{k-1}$$

Let  $x=1$

$$n 2^{n-1} = \sum_{k=1}^n k \binom{n}{k}$$

Multiply by  $x$

$$n x (1+x)^{n-1} = \sum k \binom{n}{k} x^k$$

Differentiate

$$n \left[ (1+x)^{n-1} + (n-1) \times (1+x)^{n-2} \right] = \sum_{k=1}^n k^2 \binom{n}{k} x^{k-1}$$

Let  $x=1$

$$n \left[ 2^{n-1} + (n-1) 2^{n-2} \right] = \sum_{k=1}^n k^2 \binom{n}{k}$$

or

$$n(n+1) 2^{n-2} = \sum_{k=1}^n k^2 \binom{n}{k}$$

Theorem  $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad n \geq 0$

Proof Let  $|S| = 2n$

Partition  $S$  into sets  $A$  and  $B$ ,

$$|A| = |B| = n, \quad A \cap B = \emptyset$$

Each subset  $T$  of  $S$ ,  $|T|=n$  contains  $k$  elements of  $A$  and  $n-k$  elements of  $B$

Let  $C_k$  be those  $n$  subsets which contain  $k$  elements from  $A$  (hence  $n-k$  elements from  $B$ ). Hence

$$\binom{2n}{n} = |C_0| + |C_1| + \dots + |C_n|$$

A set in  $C_k$  is gotten by taking  $k$  elements from  $A$  [ $\binom{n}{k}$  choices]

and  $n-k$  elements from  $B$

[ $\binom{n}{n-k}$  choices]. So

$$|C_k| = \binom{n}{k} \binom{n}{n-k} = \binom{n}{k} \binom{n}{k} = \binom{n}{k}^2$$

$$\text{Hence } \binom{2n}{n} = \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2$$

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2, 5, 6, 7, 8, 9, 10