

mA 416

Lesson 12

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BINOMIAL THEOREM

Theorem: Let n be a positive integer.

$$\text{Then } (x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

$$\text{Ex } (x+y)^3 = \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y^1 + \binom{3}{2} x^1 y^2 + \binom{3}{3} x^0 y^3$$

Proof. $(x+y)^n = (x+y)(x+y) \dots (x+y)$

* For $x^k y^{n-k}$, we need all ways to pick k elements x and $n-k$ elements y to get all $x^k y^{n-k}$. Thus we are choosing all combinations of k x 's and $n-k$ y 's. That is $\binom{n}{k}$ and the result holds

$$(x+y)^5 = x^5 + \binom{5}{1} x^4 y + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 + \binom{5}{4} x^1 y^4 + \binom{5}{5} x^0 y^5$$

* the coefficient of $x^k y^{n-k}$

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Theorem $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$

Proof Let $y=1$ in the binomial theorem.

Theorem $\binom{n}{k} = n \binom{n-1}{k-1}$

Proof $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

while $n \binom{n-1}{k-1} = n \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$

Theorem $\binom{n}{0} + \dots + \binom{n}{n} = 2^n$

Proof. This was shown in chapter 2

Here, set $x=1=y$ in the binomial Theorem

$$2^n = (1+1)^n = \sum_{k=0}^n \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k}$$

Theorem. $\binom{n}{0} + \binom{n}{2} + \dots = 2^{n-1}$
 $\binom{n}{1} + \binom{n}{3} + \dots = 2^{n-1}$

Proof Set $x=1$ $y=-1$ Then

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots = (1-1)^n = 0$$

Thus $\binom{n}{0} + \binom{n}{2} + \dots = \binom{n}{1} + \binom{n}{3} + \dots$

\Leftrightarrow now

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5}$$

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This last result says that for each n , the number of subsets with an even (odd) number of elements is 2^{n-1}

A combinatorial proof is instructive

Let $S = \{x_1, \dots, x_n\}$

To construct subsets:

If A is a subset

x_1 has 2 choices, in A or not

x_2 has 2 choices, in A or not

x_{n-1} has 2 choices, in A or not

Finally at x_n . ~~If~~ The choice for

x_n is already determined. If

$|A|$ is even and there are an even number of x_1, x_{n-1} in A , then x_n is not in A . If there are an odd number in A , then x_n is in A . Similar reasoning holds if $|A|$ is odd. Hence there are $2 \cdots 2^{(n-1)}$ choices = 2^{n-1}

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Computing binomials by calculus

Start with

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

Differentiate

$$\Rightarrow n(1+x)^{n-1} = \sum_{k=1}^n k \binom{n}{k} x^{k-1}$$

Let $x=1$

$$n 2^{n-1} = \sum_{k=1}^n k \binom{n}{k}$$

Multiply by x

$$n x (1+x)^{n-1} = \sum k \binom{n}{k} x^k$$

Differentiate

$$n \left[(1+x)^{n-1} + (n-1) x (1+x)^{n-2} \right] = \sum_{k=1}^n k^2 \binom{n}{k} x^{k-1}$$

Let $x=1$

$$n \left[2^{n-1} + (n-1) 2^{n-2} \right] = \sum_{k=1}^n k^2 \binom{n}{k}$$

or

$$n(n+1) 2^{n-2} = \sum_{k=1}^n k^2 \binom{n}{k}$$

Theorem $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad n \geq 0$

Proof Let $|S| = 2n$

Partition S into sets A and B ,

$$|A| = |B| = n, \quad A \cap B = \emptyset$$

Each subset T of S , $|T|=n$ contains k elements of A and $n-k$ elements of B

Let C_k be those n subsets which contain k elements from A (hence $n-k$ elements from B). Hence

$$\binom{2n}{n} = |C_0| + |C_1| + \dots + |C_n|$$

A set in C_k is gotten by taking k elements from A $\left[\binom{n}{k} \text{ choices} \right]$

and $n-k$ elements from B

$\left[\binom{n}{n-k} \text{ choices} \right]$. So

$$|C_k| = \binom{n}{k} \binom{n}{n-k} = \binom{n}{k} \binom{n}{k} = \binom{n}{k}^2$$

$$\text{Hence } \binom{2n}{n} = \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2$$

The Multinomial Theorem

Consider $(x_1 + \dots + x_k)^n$.

$$(x_1 + \dots + x_k)^n = (x_1 + x_k) \cdots (x_1 + x_k)$$

When multiplying, one x_i is picked from each factor giving term

$$x_1^{n_1} \cdots x_k^{n_k}$$

$$\text{where } n_1 + n_k = n$$

The coefficient is the number of ways we can take an element from each factor, $n_1 x_1$'s, $n_2 x_2$'s ... x_k 's. So pick $\binom{n}{n_1} x_1$,

$$\binom{n-n_1}{n_2} x_2 \text{'}s \cdots \binom{n-n_1-\dots-n_{k-1}}{n_k} x_k \text{'}s$$

The number of ways is

$$\binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-\dots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1! n_2! \cdots n_k!}$$

Hence

$$(x_1 + x_2 + \dots + x_k)^n = \sum \frac{n!}{n_1! n_2! \dots n_k!} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

$n_1 + n_2 + \dots + n_k = n$

The sum is over all k tuples (n_1, \dots, n_k) such that $n_1 + n_2 + \dots + n_k = n$

Ex. In $(x_1 + x_2 + x_3)^{10}$ what is the term with monomial $x_1^4 x_2^5 x_3^1$
 Ans $\frac{10!}{4! 5! 1!}$

We write $\frac{n!}{n_1! n_2! \dots n_k!}$ as $\binom{n}{n_1, \dots, n_k}$

where $n_1 + \dots + n_k = n$

Ex Compute $(x+y+z)^3 =$

$$\binom{3}{3,0,0} x^3 + \binom{3}{0,3,0} y^3 + \binom{3}{0,0,3} z^3 +$$

$$\binom{3}{2,1,0} x^2 y + \binom{3}{2,0,1} x^2 z + \binom{3}{0,2,1} y^2 z +$$

$$\binom{3}{1,2,0} x y^2 + \binom{3}{1,0,2} x z^2 + \binom{3}{0,1,2} y z^2$$

$$+ \binom{3}{1,1,1} x y z$$

$$\text{There } \binom{3}{3,0,0} = \frac{3!}{3!0!0!} = 1$$

$$\binom{3}{2,1,0} = \frac{3!}{2!1!0!} = 3$$

$$\binom{3}{1,1,1} = \frac{3!}{1!1!1!} = 6$$

Ex What is the coefficient of x^3y^2z in $(2x+5y+z)^6$:

$$8 \times 2^3 5^2 z \binom{6}{3,2,1} = (400) \frac{6!}{3!2!1!} x^3 y^2 z$$

Ex How many different terms appear in $(x+y+z)^6$: Ans: solutions to

$$n_1 + n_2 + n_3 = 6 \quad n_1 \geq 0 \quad n_2 \geq 0 \quad n_3 \geq 0 \quad \text{or}$$

$$\binom{6+3-1}{6}$$

Pascal's formula in this setting is

$$\binom{n}{n_1 \dots n_k} = \binom{n-1}{n_1-1 \ n_2 \dots n_k} + \binom{n-1}{n_1 \ n_2-1 \dots n_k} + \dots$$

$$\binom{n-1}{n_1 \ n_2 \ \dots \ n_{k-1}}$$

Proof. The right hand side equals

$$\frac{(n-1)!}{(n-1)! \dots (n_k)!} + \frac{(n-1)!}{n_1! (n_2-1)! \dots n_k!} + \dots + \frac{(n-1)!}{n_1! \dots (n_{k-1})! (n_k-1)!}$$

= Multiply the first term by $\frac{n_1}{n_1}$ etc
to get

$$\frac{n_1(n-1)!}{n_1! \dots n_k!} + \frac{n_2(n-1)!}{n_1! n_2! \dots n_k!} + \dots + \frac{n_k(n-1)!}{n_1! \dots (n_{k-1})! \cancel{n_k!}}$$

$$= \frac{(n_1 + \dots + n_k)(n-1)!}{n_1! \dots n_k!} = \frac{n(n-1)!}{n_1! \dots n_k!}$$

$$= \frac{n!}{n_1! \dots n_k!}$$

Problems Page 154-159

2, 5, 6, 7, 8, 9, 10

13, 14, 15, 16, 18, 23, 24

37, 39, 40