

MA 416

Lesson 12

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Recall:  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  where

$k$  and  $n$  are non negative integers

We generalize this by letting  $n$

be any real number!

$$\binom{n}{k} = \begin{cases} \frac{n(n-1)\dots(n-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k \leq -1 \end{cases}$$

$$\text{Ex } \binom{3/2}{3} = \frac{3/2 \cdot 1/2 \cdot (-1/2)}{3 \cdot 2 \cdot 1}$$

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3 \cdot 2 \cdot 1}$$

$$\binom{3}{-2} = 0$$

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The formulas

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{and}$$

$$k \binom{n}{k} = n \binom{n-1}{k-1} \quad \text{remain valid.}$$

They can be shown by using the definition on the preceding page

Iteration of Pascal's formula gives new results

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\ &= \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2} \\ &= \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-3}{k-2} + \binom{n-3}{k-3} \\ &= \dots \\ &= \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-3}{k-2} + \dots \\ &\quad \binom{n-k}{1} + \binom{n-k-1}{0} + \binom{n-k-1}{-1} \end{aligned}$$

The last term is 0 and we stop

Replace  $n$  by  $n+k+1$  to get

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$$\binom{n+k+1}{k} = \binom{n+k}{k} + \binom{n+k-1}{k-1} + \dots + \binom{n+1}{1} + \binom{n}{0}$$

Another identity comes by iterations on the first term. Assume  $n$  is an integer

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\ &= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-1}{k-1} \\ &= \binom{n-3}{k} + \binom{n-3}{k-1} + \binom{n-2}{k-1} + \binom{n-1}{k-1} \\ &= \dots \\ &= \binom{0}{k} + \binom{0}{k-1} + \dots + \binom{n-1}{k-1} \end{aligned}$$

$\binom{0}{k} = 0$ . Replace  $n$  by  $n+1$ ;

$$\binom{n+1}{k} = \binom{0}{k} + \binom{1}{k} + \dots + \binom{n}{k}$$

The first non zero term

is  $\binom{k}{k} = 1$

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Unimodal.

In any row in Pascal's  $\Delta$   
 the elements increase and then  
 they decrease. Such a sequence  
 is called unimodal. That is

$S_0, S_1, \dots, S_n$  is unimodal if

$$S_0 \leq S_1 \leq \dots \leq S_{j-1} \leq S_j \geq S_{j+1} \geq \dots \geq S_n$$

Theorem  $\binom{n}{0}, \dots, \binom{n}{n}$  is unimodal

In fact

If  $n$  is even

$$\binom{n}{0} < \binom{n}{1} < \dots < \binom{n}{n/2} > \binom{n}{n/2+1} > \dots > \binom{n}{n}$$

If  $n$  is odd

$$\binom{n}{0} < \binom{n}{1} < \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2} > \binom{n}{(n+3)/2} > \dots > \binom{n}{n}$$

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Proof Consider

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}} = \frac{n-k+1}{k}$$

Hence  $\binom{n}{k-1} < \binom{n}{k} \iff k < n-k+1$

$$\binom{n}{k-1} = \binom{n}{k} \iff k = n-k+1$$

$$\binom{n}{k-1} > \binom{n}{k} \iff k > n-k+1$$

Now  $k < n-k+1 \iff k < \frac{n+1}{2}$

If  $n$  is even  $k < \frac{n+1}{2} \iff k \leq \frac{n}{2}$

If  $n$  is odd  $k < \frac{n+1}{2} \iff k \leq \frac{n-1}{2}$

Hence the coefficients increase as stated

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$$\text{Now } k = n + k + 1 \iff 2k = n + 1$$

$$\text{If } n \text{ is even } \quad 2k \neq n + 1$$

$$\text{If } n \text{ is odd } \quad 2k = n + 1 \iff k = \frac{n+1}{2}$$

Thus if  $n$  is even, The coefficients are never equal

If  $n$  is odd they are equal exactly when

$$\binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}$$

The second half of the sequence, where the terms are decreasing, is shown in a similar matter

## 7 Chains and Antichains

Let  $S$  be a set,  $|S|=n$ .

An antichain of  $S$  is a collection of subsets, no two of which have containment; i.e. if  $A$  and  $B$  are any two of the antichain

$$A \not\subseteq B \text{ and } B \not\subseteq A.$$

Ex let  $T$  be all subsets of  $S$  of the same size. Then this is an antichain since different sets of the same size are not contained in each other

By the previous work on sequences of binomial coefficients, if  $n$  is even there is an antichain of  $\binom{n}{n/2}$  elements and if  $n$  is odd

There is an antichain of  $\binom{n}{\lfloor \frac{n-1}{2} \rfloor}$  elements and one of  $\binom{n}{\lfloor \frac{n+1}{2} \rfloor}$  elements.

For  $|S|=n$ , these are the largest antichains that  $S$  can have

By the last Theorem, these are the largest antichains whose sets have the same size. Could there be a larger antichain where the sets have different sizes?

The answer is no as we will now see.

Example  $S = \{a, b, c, d, e\}$

Antichain:  $\left[ \begin{array}{l} \{a, b\} \{a, c\} \{a, d\} \{a, e\} \{b, c\} \\ \{b, d\} \{b, e\} \{c, d\} \{c, e\} \{d, e\} \end{array} \right] = A$

$$|A| = 10 = \binom{5}{2}$$

All subsets with 3 elements also has 10 elements

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Another concept is that of a chain. Let  $S$  be a set,  $|S| = n$ . A chain<sup>c</sup> from  $S$  is a collection of subsets of  $S$  such that if  $A, B \in C$ , then either  $A \subset B$  or  $B \subset A$ .

$$G \times S = \{a, b, c, d\}$$

$$C = \{ \emptyset, \{a, c\}, \{a, b, c, d\} \}$$

$$C = \{ \emptyset, \{d\}, \{c, d\}, \{b, c, d\}, S \}$$

The second example is a maximal chain in that no more subsets could be included. It contains one subset of each possible order. Thus  $|C| = 5$ . Generally, if  $|S| = n$  the order of a maximal chain is  $n+1$ . The number of maximal chains is  $n!$

Note that if  $A$  is an antichain of  $S$  and  $C$  is a chain of  $S$ , then  $|A \cap C| = 0$  or  $1$ .  
 For if  $A, B \in C$ , then  $A \subseteq B$  or  $B \subseteq A$ . If  $A, B \in A$ ,  $A \not\subseteq B$  and  $B \not\subseteq A$ . Thus there can not be two such elements.

Theorem. Let  $|S| = n$ . Then an antichain has at most

$\binom{n}{n/2}$  elements if  $n$  is even

$\binom{n}{n-1/2} = \binom{n}{n+1/2}$  elements if  $n$  is odd

Proof. Let  $A$  be an antichain

Consider  $(B, C)$  where  $B \in A$

and  $C$  is a maximal chain

containing  $B$ . Any given  $C$  contains

at most one  $B$ . Let

$\beta =$  number of the ordered  
 Pairs  $(B, C)$

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Since the number of possible  $C$  is  $n!$  and there is at most one  $B$  for each  $C$ ,

$$\beta \leq n!$$

Consider some  $B \in A$  If  $|B| = k$ , there are at most  $k!(n-k)!$  maximal antichains  $C$  containing  $B$  let  $\alpha_k$  be

the number of subsets in  $A$  of size  $k$ . Hence  $|A| = \sum_{k=0}^n \alpha_k$

$$\text{So } \beta = \sum \alpha_k k!(n-k)!$$

BUT  $\beta \leq n!$  so

$$\sum \alpha_k k!(n-k)! \leq n!$$

$$\rightarrow \sum \alpha_k \frac{k!(n-k)!}{n!} \leq 1$$

$$\rightarrow \sum \frac{\alpha_k}{\binom{n}{k}} \leq 1 \rightarrow \sum_{k=0}^n \alpha_k \leq \binom{n}{k}$$

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$$\rightarrow \text{IPI } \beta \leq \binom{n}{k}$$

Then  $\binom{n}{k} \leq \binom{n}{n/2}$  if  $n$  is even

$\binom{n}{k} \leq \binom{n}{(n-1)/2}$  if  $n$  is odd

which gives the result.

Problems Ch. 5

13, 14, 15, 16, 18