

M A 416

Lesson 15

More On Posets

More on anti chains and chains
in $P(X)$, the subsets of set X
and relation \subseteq . We consider

a certain partition of $P(X)$ in terms
of a certain type of chain

Recall, if $|X|=n$, then the
largest anti chain has size $\binom{n}{\lfloor n/2 \rfloor}$

where $\lfloor n/2 \rfloor$ is the largest integer
less than or equal to $n/2$

$$\text{Ex } \binom{5}{\lfloor 5/2 \rfloor} = \binom{5}{2} = 10$$

$$\binom{6}{\lfloor 6/2 \rfloor} = \binom{6}{3} = 20$$

Then $P(X)$ can be partitioned into
that many chains, each of which
intersects the maximal antichain
in exactly one set.

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Def. A chain partition of the subsets of $X, |X|=n$, is called symmetric provided that

(a) Each subset in a chain in the partition has one more element than the preceding set in the chain

(b) In a chain, the size of the first subset + the size of the last subset equals n . If there is just one subset in the chain, then the size is $\lceil \frac{n}{2} \rceil$ (n is even in this case). Each chain is a symmetric chain partition contains exactly one $\lceil \frac{n}{2} \rceil$ subset

We show how to construct a symmetric chain partition for each n .

For a given n there are a certain number of chains. To get the $n+1$ case, do two type of steps

- (i) To each chain in the n case, add a new set obtained by inserting $n+1$ in the first set in the chain
- (ii) Remove the final set in the chain and insert $n+1$ in the remaining sets in the chain

$$\text{Ex } n=1 \quad \emptyset \subset \{1\}$$

$$n=2 \quad \begin{array}{l} \text{a. } \emptyset \subset \{1, 2\} \subset \{1\} \cup \{2\} \\ \text{b. } \{1, 2\} \end{array}$$

$$n=3 \quad \begin{array}{l} \text{a. } \emptyset \subset \{1, 2, 3\} \subset \{1, 2\} \cup \{3\} \\ \quad \quad \quad \{1, 3\} \cup \{2, 3\} \\ \text{b. } \{1, 3\} \subset \{1, 2, 3\} \end{array}$$

$$n=4 \quad \begin{array}{l} \text{a. } \emptyset \subset \{1, 2, 3, 4\} \subset \{1, 2, 3\} \cup \{4\} \subset \{1, 2\} \cup \{3, 4\} \\ \quad \quad \quad \{1, 3\} \cup \{2, 3\} \cup \{1, 4\} \cup \{2, 4\} \\ \quad \quad \quad \{1, 3, 4\} \subset \{1, 2, 3, 4\} \\ \text{b. } \{1, 3, 4\} \subset \{1, 2, 3, 4\} \end{array}$$

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With the n case completed, following the two steps allows every subset to appear exactly once.

Every subset with the elements $1, \dots, n$ appear in the n -case and in step a they are listed.

The sets with $n+1$:

Every final step set in a chain has $n+1$ inserted in it to get a new final set in step a.

In Step b, ~~this new set~~ every set from the n case has $n+1$ inserted except the last set which is removed. The removal is done because if it is not removed inserting $n+1$ would have the set appear twice, at the end of the chain in step a and at the end of the chain in step b. So removal is necessary.

Newton's Binomial Theorem.

Recall the binomial theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

where n is a positive integer.

Newton generalized it for
 $n = 2$ any real number:

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}$$

where $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$

Ex $(x+y)^{5/2} = x^0 y^{5/2} + \binom{5/2}{1} x^1 y^{3/2} + \binom{5/2}{2} x^2 y^{1/2} + \binom{5/2}{3} x^3 y^{-1/2} \dots$

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$$(z+1)^k = \sum_{k=0}^{\infty} (\text{coefficient}) z^k \quad \text{follows.}$$

$$\text{Let } k = -n$$

$$\binom{k}{n} = \binom{-n}{k} = \frac{-n(-n-1) \cdots (-n-k+1)}{k!}$$

$$= (-1)^k \frac{(n)(n+1) \cdots (n+k-1)}{k!}$$

$$= (-1)^k \binom{n+k-1}{k} \quad \text{Therefore}$$

$$\rightarrow (z+1)^{-n} = \frac{1}{(z+1)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} z^k$$

When $n = 1$,

$$\frac{1}{z+1} = \sum_{k=0}^{\infty} (-1)^k z^k$$

~~when~~

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

$\text{Let } \alpha = \sqrt{z} \quad (\text{Square roots})$

$$\left(\frac{\alpha}{z}\right) = \left(\frac{\sqrt{z}}{z}\right) = \frac{1}{2} \underbrace{(-\frac{1}{2}-1) \dots (-\frac{1}{2}-k+1)}_{k!}$$

$$= \frac{1}{2} \left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) \dots \left(-\frac{-2k+3}{2} \right)$$

$$= \frac{1}{2^k} \frac{(-1)^{k-1}}{k!} \frac{1}{1} \frac{3}{3} \frac{5}{5} \dots \frac{2k-3}{2k-3}$$

$$= \frac{1}{2^k} \frac{(-1)^{k-1}}{k!} \frac{1}{2} \frac{2}{4} \frac{3}{5} \frac{4}{6} \dots \frac{2k-3}{2k-2} \frac{2k-2}{2k-1}$$

$$= \frac{1}{2^k} \frac{(-1)^{k-1}}{k!} \frac{(2k-2)!}{2^{k-1} (k-1)!}$$

$$= \frac{1}{2^k} \frac{1}{k} \frac{(-1)^{k-1}}{2^{k-1}} \frac{(2k-2)!}{(k-1)!}$$

$$= \frac{1}{k} \frac{(-1)^{k-1}}{2^{2k-1}} \binom{2k-2}{k-1}$$

\Leftarrow

$$\sqrt{z+1} = (z+1)^{\frac{1}{2}} = \begin{cases} 1 \\ \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k \end{cases}$$

$$= 1 + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k 2^{2k-1}} \binom{2k-2}{k-1} z^k$$

$$\begin{aligned}
 \text{Ex } \sqrt{20} &= \sqrt{4+16} = 4\sqrt{.25+1} \\
 &= 4 \left[1 + \frac{1}{2} (.25) - \frac{1}{8} (.25)^2 + \frac{1}{16} (.25)^3 \right] \\
 &= 4.472
 \end{aligned}$$

Problems Chapter 5

33, 46, 47

In 47 you do not need

the expansion past

several terms. So you
do not need the detail
as in ^{the} square root
derivation.