

M A 416

Lesson 16

Let S be a set with subsets A and B .

Notation $S - A = \bar{A} = \text{set of elements in } S \text{ that are not in } A.$

Several results:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

* $|A \cup B| = |A| + |B| - |A \cap B|$

This well known last result affords the opportunity to introduce a proof technique

If $x \in \text{set } C$ assign 1

$x \notin \text{set } C$ assign 0

For *

• If $x \in A, x \in B$ *

$$1 = 1 + 1 - 1$$

• If $x \in A, x \notin B$ *

$$1 = 1 + 0 - 0$$

Same if $x \notin A, x \in B$

If $x \notin A, x \notin B,$

$$0 = 0 + 0 - 0$$

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We set identities each time letting
 x run through S gives that Δ holds

Combining several results on the
 preceding page gives

$$|S| - |A_1 \cup A_2| = |\overline{A_1 \cup A_2}| = |\bar{A}_1 \cap \bar{A}_2|$$

and

$$|\bar{A}_1 \cap \bar{A}_2| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|$$

Ex Among 100 people,
 60 like coffee
 50 like tea
 40 like both

How many like neither?

$$A_1 = \{\text{likes coffee}\}$$

$$A_2 = \{\text{likes tea}\}$$

$$A_1 \cap A_2 = \{\text{likes both}\}$$

$$\text{Likes neither} = |\overline{A_1 \cup A_2}| = |\bar{A}_1 \cap \bar{A}_2|$$

$$= |S| - |A_1| - |A_2| + |A_1 \cap A_2|$$

$$= 100 - 60 - 50 + 40 = 30$$

Also ~~$|S| - |A_1 \cap A_2| = |\overline{A_1 \cup A_2}|$~~

The result generalizes to the inclusion-exclusion principle:

Theorem. If A_1, \dots, A_m are subsets of S , then

$$|\bar{A}_1 \cap \dots \cap \bar{A}_m| = |S| - \sum |A_i|$$

$$+ \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| \dots$$

$$+ (-1)^m |A_1 \cap \dots \cap A_m|$$

where each sum is over all possible combination of the subsets

so the number of sets in

$$\sum |A_i| \text{ is } \binom{m}{1}$$

$$\sum |A_i \cap A_j| \text{ is } \binom{m}{2}$$

$$\sum |A_i \cap A_j \cap A_k| \text{ is } \binom{m}{3}$$

$$\sum |\text{or } A_i| = \binom{m}{k}$$

$$\sum |A_1 \cap \dots \cap A_m| = \binom{m}{m}$$

When $m=3$, the inclusion-exclusion principle is

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - (|A_1| + |A_2| + |A_3|) \\ + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ - |A_1 \cap A_2 \cap A_3|$$

Using the 0,1 technique:

If x is in no A_i

$$1 = 1 - 0 + 0 -$$

If x is in one A_i

$$0 = 1 - 1 + 0$$

If x is in two A_i

$$0 = 1 - 2 + 1 + 0 -$$

If x is in k of the A_i

$$0 = 1 - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} \cdot (-1)^k \binom{k}{5}$$

We saw in the last chapter
that this is an identity

Hence the result holds.

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Corollary

$$\begin{aligned} |A_1 \cup \dots \cup A_m| &= |S| - |\bar{A}_1 \cap \dots \cap \bar{A}_m| \\ &= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \\ &\quad + \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_m| \end{aligned}$$

Ex Find the number of integers between 1 and 1000 not divisible by 5, 6 and 8

$$S = \{1, \dots, 1000\} \quad |S| = 1000$$

A_1 = {integers divisible by 5}

A_2 = {integers divisible by 6}

A_3 = {integers divisible by 8}

\bar{A}_1 not divisible by 5

\bar{A}_2 not divisible by 6

\bar{A}_3 not divisible by 8

$\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$ not divisible by 5, 6, 8

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - (|A_1| + |A_2| + |A_3|)$$

$$+ (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)$$

$$- |A_1 \cap A_2 \cap A_3|$$

$$|A_1| = \frac{1000}{5} = 200$$

$$|A_2| = \frac{1000}{6} = 166$$

$$|A_3| = \frac{1000}{8} = 125$$

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$A_1 \cap A_2$ divisible by 30

$A_1 \cap A_3$ divisible by 40

$A_2 \cap A_3$ divisible by 24 (both 6 and 8
so 24)

$A_1 \cap A_2 \cap A_3$ divisible by 120

$$|A_1 \cap A_2| = \left\lfloor \frac{1000}{30} \right\rfloor = 33$$

$$|A_1 \cap A_3| = \left\lfloor \frac{1000}{40} \right\rfloor = 25$$

$$|A_2 \cap A_3| = \left\lfloor \frac{1000}{24} \right\rfloor = 41$$

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{1000}{120} \right\rfloor = 8$$

$$|\tilde{A}_1 \cap \tilde{A}_2 \cap \tilde{A}_3| = 1000 - (200 + 166 + 725)$$

$$+ (33 + 25 + 41) - 8$$

$$= 600$$

Example How many permutations of
MATH IS FUN

are there where MATH or IS or
FUN DO NOT OCCUR in the

expression

$$S = \{ m, A, T, H, I, S, F, U, N \}$$

$$|S| = 9!$$

A_1 = permutations with MATH

A_2 = permutations with IS

A_3 = permutations with FUN

A_3 = permutations with one symbol

In A_1 treat MATH as one symbol

$$|A_1| = 6!$$

In A_2 treat IS as one symbol

$$|A_2| = 8!$$

In A_3 treat FUN as one symbol

$$|A_3| = 7!$$

We are looking for
 $|A_1 \cap A_2 \cap A_3|$

$$\begin{aligned}
 |A_1 \cap A_2 \cap A_3| &= 15! - (|A_1| + |A_2| + |A_3|) \\
 &\quad + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\
 &\quad - |A_1 \cap A_2 \cap A_3|
 \end{aligned}$$

In $A_1 \cap A_2$ there is one symbol and
 I_3 is one symbol

$$|A_1 \cap A_2| = 5!$$

Also

$$|A_1 \cap A_3| = 4!$$

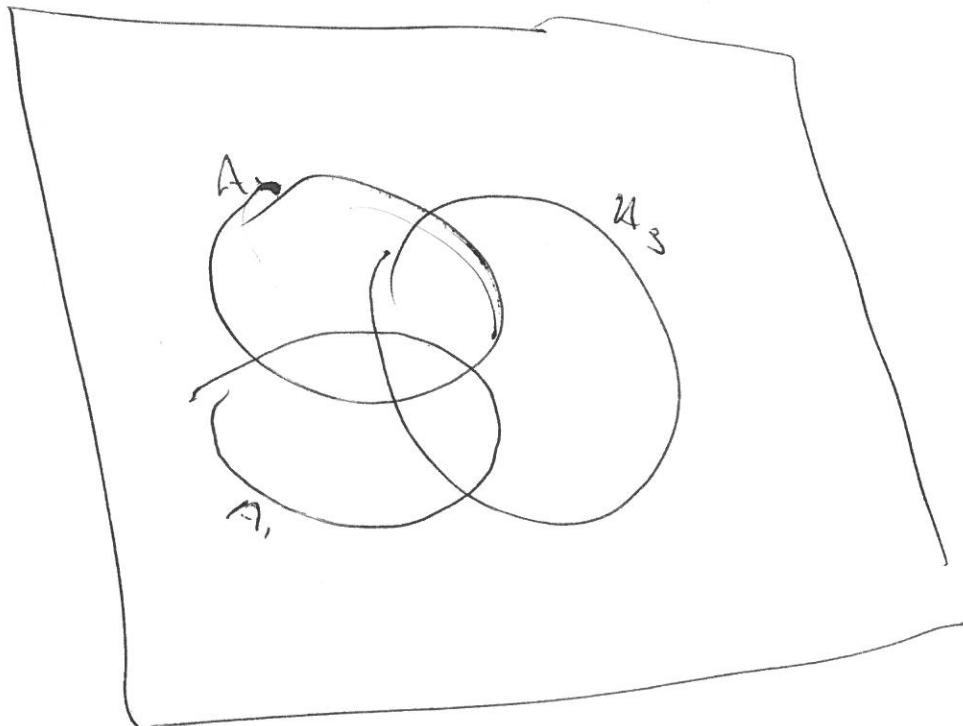
$$|A_2 \cap A_3| = 6!$$

$$|A_1 \cap A_2 \cap A_3| = 3!$$

The inclusion-exclusion principle gives

$$\begin{aligned}
 9! - (6! + 8! + 7!) + (4! + 6! + 5!) \\
 - 3!
 \end{aligned}$$

For 3 subsets the ideas can be seen geometrically



$$\begin{aligned}
 |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\
 &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\
 &\quad + |A_1 \cap A_2 \cap A_3|
 \end{aligned}$$

We add all elements from A_1, A_2, A_3 , then need to subtract the intersections and then add back the central intersection which has been added 3 times then subtracted 3 times, so we need to add it back.

Consider the special case when for sets A_1, \dots, A_n , we have

$$|A_i| = \alpha_i \text{ for each } i$$

$$|A_{i_1} \cap A_{i_2}| = \alpha_2 \text{ for each } i_1 \neq i_2$$

$$|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = \alpha_3 \text{ for each } \{i_1, i_2, i_3\}$$

$$|A_1 \cap \dots \cap A_n| = \alpha_n \text{ and}$$

$$|\mathcal{S}| = \alpha_0$$

Then

$$\begin{aligned} |\bar{A}_1 \cap \dots \cap \bar{A}_n| &= \alpha_0 - \alpha_1 \binom{1}{1} + \alpha_2 \binom{1}{2} \\ &\quad \cdots \alpha_n (-1)^n \binom{n}{n} \end{aligned}$$

This follows directly from
the inclusion-exclusion
Theorem

Ex How many integers from 0 to 99999 have among their digits each of 2, 5, 8

S = all 5 digit numbers

$$A_1 = \text{no } 2$$

$$A_2 = \text{no } 5$$

$$A_3 = \text{no } 8$$

$$|A_1| = |A_2| = |A_3| = 9^5 = \alpha_1$$

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = 8^5 = \alpha_2$$

$$|A_1 \cap A_2 \cap A_3| = 7^5 = \alpha_3$$

$$|S| = 10^5$$

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= |S| - \alpha_1 \binom{3}{1} + \alpha_2 \binom{3}{2} - \alpha_3 \binom{3}{3} \\ &= 10^5 - 9^5 \cdot 3 + 8^5 \cdot 3 - 7^5 \cdot 1 \end{aligned}$$

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