

Lesson 5

Combinations of Multisets

Recall that last time we discussed permutations of multisets and found the following: If S is a multiset with objects of k types, a_1, \dots, a_k and $S = \{n_1 a_1, \dots, n_k a_k\}$, $n = n_1 + \dots + n_k$, then the number of permutations of equals

$$\frac{n!}{n_1! \cdots n_k!}$$

If $k=2$, this equals

$$\frac{n}{n_1!(n-n_1)!} = \binom{n}{n_1}$$

We now consider combinations of multisets. An r combination of multiset S is an unordered selection of r of the elements

Ex $S = \{2a, 1b, 3c\}$. Let $r=3$
 The 3-combinations are
 $\{2a, c\}, \{2a, b\}, \{a, b, c\}, \{a, 2c\}$
 $\{2a, b\}, \{2c\}, \{3c\}$
 $\{b, 2c\}$

Thm. Let $S = \{\omega q_1, \dots, \omega q_k\}$. The number of r -combinations of S is $\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$

So we are counting the number of sets $\{x_1 q_1, \dots, x_k q_k\}$ where $x_j \geq 0$ for each j and $x_1 + \dots + x_k = r$.

Thus we are looking for the number of solutions to

$$x_1 + \dots + x_k = r, \text{ each } x_i \in \text{non-meg. integer}$$

To find the answer, we form another problem. Consider the multiset

$$T = \{r y, (k-1) z\}$$

How many permutations are there? This is part of what we started with today!

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

Consider these permutations by

1. putting the z 's in a row

$\begin{matrix} & & & & \\ z & z & z & z & z \end{matrix}$

The y 's fill in the spaces between the z 's, including between the first z and in front of the last z . The following y 's in each space is number of y 's in each space is a sequence x_1, \dots, x_k

like $y = 445 = 45+444+444$

Here $x_1=1$ $x_2=3$ $x_3=2$ $x_4=4$ $x_5=3$

The x_i add to r :

$$x_1 + \dots + x_k = r$$

Any sequence of x 's that add to r works. BUT This is exactly the r -combination problem we started with. Hence the answer to the r -combination problem is $\binom{r+k-1}{r}$.

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Ex. (A good one). A donut shop makes 8 kinds of donuts

How many different boxes of donuts (12 in a box) can be made

The multiset is made up of

The donuts $S = \{x_1, a_1, \dots, x_8, a_8\}$

and $r=12$. We want the number of solutions to

$$x_1 + \dots + x_8 = 12 \quad x_i \geq 0$$

$$\text{Ans } \binom{12+8-1}{8-1} = \binom{19}{12} = \binom{19}{7}$$

Ex. Find the number of solutions

$$\text{to } x_1 + x_5 = 15 \quad x_i \text{ non neg integers}$$

$$k=5 \quad r=15 \quad \text{Ans } \binom{15+4}{15} = \binom{19}{11}$$

17 substitution allows us to
Solve: Find the number of
solutions to

$$x_1 + x_2 + x_3 + x_4 = 20$$

$$x_1 \geq 2 \quad x_2 \geq 1 \quad x_3 \geq 3 \quad x_4 \geq 0$$

$$\text{Let } y_1 = x_1 - 2 \quad y_3 = x_3 - 3$$

$$y_2 = x_2 - 1 \quad y_4 = x_4 - 0$$

Then substitution gives

$$y_1+2 + y_2+1 + y_3+3 + y_4 = 20$$

$$y_1 + y_2 + y_3 + y_4 = 14 \quad y_i \geq 0$$

$$\rightarrow \text{Ans} \left(\begin{matrix} 14 \\ 14 \end{matrix} \right) = \left(\begin{matrix} 14+3 \\ 3 \end{matrix} \right)$$

These ideas can be applied to problems that are a little different

Ex. Find the number of nondecreasing sequences of length r whose terms come from $1, 2, \dots, k$.

$$\text{Ex. } r=8 \quad k=3$$

$$1, 1, 2, 2, 2, 3, 3, 3 \quad \text{or}$$

$$1, 1, 1, 1, 1, 3, 3, 3 \quad \text{etc}$$

The multiset is $\{\infty 1, \infty 2, \dots, \infty k\}$
and we are picking r of them.

i.e. x_1 's, x_2 's, x_3 's, ..., x_k 's are

$$x_1 + x_2 + \dots + x_k = r \quad x_i \geq 0$$

This is our problem and answer

$$\left(\begin{matrix} r+k-1 \\ r \end{matrix} \right)$$

Example 20 books are to be placed on 5 shelves, each which will hold 20 books

(a) How many ways can this be done where we are asking how many ways counting the number of books on each shelf

Ans Let x_j be the number of books on shelf j . Then $x_1 + x_2 + x_3 + x_4 + x_5 = 20$,
 $x_i \geq 0$. Ans $\binom{20+4}{4}$

(b) How many ways if we count the ways the books are on the shelves

Ans Each of the 20 books has 5 choices. So 5^{20}

(c) How many ways if we count the order of the books on the shelves

Ans First count the numbers as in part (a). $\binom{24}{4}$. Then the first book has 20 places possible, the second has 19, ..., Total $20!$. This is independent of the numbers

Ans $\binom{24}{4} \cdot 20!$ on the shelves

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24, 25, 28, 31, 32, 35

37, 38, 39, 41, 42