

Lesson 6

Finite Probability

Finite Probability

The idea: There is an experiment with a set of outcomes, each equally likely. The set of all outcomes is called the sample space. If there are n outcomes, then each one has a $\frac{1}{n}$ chance of occurring. We say the probability of an event occurring is $\frac{1}{n}$. Notation: If s is the outcome, then $\Pr(s) = \frac{1}{n}$

An event is any subset of the sample space

Ex A die is tossed. The sample space is $\{1, 2, 3, 4, 5, 6\} = S$
If $s \in S$, then $\Pr(s) = \frac{1}{6}$ let E be the event that the number is odd
Then $\Pr(\text{event}) = \frac{1}{2}$.

Generally, if E is an event in S , then

$$\Pr(E) = \frac{|E|}{|S|}. \quad \text{Clearly } 0 \leq \Pr(E) \leq 1$$

with $\Pr(E) = 0 \leftrightarrow E$ is the empty set

$$\Pr(E) = 1 \leftrightarrow E = S$$

Ex. Let 2 die be rolled and record the result as an ordered pair with the designated die's outcome listed first. So $S = \{(a,b) \mid 1 \leq a,b \leq 6\}$

An event could be that the sum of the pair is 4,

$$E = \{(1,3), (2,2), (3,1)\} \quad |E| = 3$$

Since $|S| = 6 \cdot 6 = 36$, $Pr(E) = 3/36$

Ex. Choose a sequence of ⁿ positive integers between 1 and n:

$$\{L_1, \dots, L_n\}$$

$|S| = n^n$ as each of the n integers can be in each place. $|E| = n!$ as we set $n \cdot (n-1) \cdot \dots \cdot (1)$ in the n places

$$Pr(E) = \frac{n!}{n^n}$$

Ex 5 rooks are placed in non attacking positions on an 8x8 board. What is

the probability the rooks are in rows 1, ..., 5 and columns 4, ..., 8.

To find the number of elements

In the sample space S , take 5 out of 8 rows in $\binom{8}{5}$ ways, 5 out of 8 columns in $\binom{8}{5}$ ways and assign the 5 row to the 5 columns in $5!$ ways. So

$$|S| = \binom{8}{5} \binom{8}{5} 5!$$

If E is the event the rooks are in the picked 5 rows + 5 columns and there are $5!$ ways of doing it

$$\text{So } Pr(E) = \frac{5!}{\binom{8}{5} \binom{8}{5} 5!}$$

POKER

POKER is very good to use as examples in probability. A hand is a 5-card hand and the number of hands is $\binom{52}{5}$. The hand are S and $|S| = \binom{52}{5}$

There are a number of hands which have names in poker.

We find the probability of getting several of these named hands

a Full House: 3 cards of one ^{rank} ~~suit~~ and 2

two of another. There are 13 ranks and we pick 3 out of 4 cards at that rank in $\binom{4}{3}$ ways. There are 12 picks for the second rank, so 12 ranks and $\binom{4}{2}$ ways to get the cards

$$13 \binom{4}{3} \binom{4}{2} 12 = |E|$$

$$\text{So } \Pr(E) = \frac{13 \cdot 12 \binom{4}{3} \binom{4}{2}}{\binom{52}{5}}$$

b Flush: 5 cards ~~as a~~ of same suit.

$|E|$ found by picking suit (4 ways)

and the 5 cards $\binom{13}{5}$. $|E| = 4 \binom{13}{5}$

$$\Pr(E) = \frac{|E|}{|S|} = \frac{4 \binom{13}{5}}{\binom{52}{5}}$$

c. Straight. 5 cards of each rank differing by 1. In poker, ace counts as 1 and as 14 (one after king) so there are 10 sequences that increase by 1 in each step and 4 choices for each suit in the 5 numbers: 4^5 $|E| = 10 \cdot 4^5$

$$Pr(E) = \frac{10 \cdot 4^5}{\binom{52}{5}}$$

d. \rightarrow Straight flush. Now the cards must be a straight of the same suit: $4 \cdot 10 = |E|$

$$Pr(E) = \frac{4 \cdot 10}{\binom{52}{5}}$$

e. Two pairs.

Pick 2 pairs in $\binom{13}{2}$ ways

Now each of these can be pairs in $\binom{4}{2}$ ways

Now pick remaining card in 44 ways (it can't be of the 2 suits used in the pairs)

$$|E| = \binom{13}{2} \binom{4}{2} \binom{4}{2} 44$$

$$Pr(E) = \frac{\binom{13}{2} \binom{4}{2} \binom{4}{2} 44}{\binom{52}{5}}$$

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Ex. a. 5 rooks can be placed in a particular 5 row and 5 columns in a 8×8 chessboard in $\binom{8}{5} \binom{8}{5}$ ways

b. If they are also non-attacking the placements can be done

in $\binom{8}{5} \binom{8}{5} 5!$ ways

c. The event that they are in

the rows 1 2 3 4 5 and columns

4 5 6 7 8 can be done in $5!$ ways

$$\text{Hence } \Pr(E) = \frac{5!}{\binom{8}{5} \binom{8}{5} 5!} = \frac{1}{\left(\frac{8!}{5!3!}\right)^2}$$

Problems p 67 :

56, 57, 62, 63

Problem 20 p. 65

a. 20 sticks in a row
6 are to be choosen.
How many ways?

$$n_1 \mid n_2 \mid n_3 \mid n_4 \mid n_5 \mid n_6 \mid n_7$$

The 6 choosen sticks have non-choosen sticks between them, and at the beginning and end. Suppose n_1 is the number at the beginning, n_2 the number between stick 1 and stick 2 etc.

Then $n_1 + \dots + n_7 = 14$ $n_i \geq 0$ all i

Ans $\binom{14 + (7-1)}{14}$

b and c have slight variations