

MA 416

Lesson 7

1 Pigeonhole Principle

Theorem. Given $n+1$ objects and n boxes, then at least 2 objects would go in the same box.

Proof If each box contains at most 1 object, this accounts for $\leq n$ objects. This is not enough so the result holds.

Ex: $n+1$ pigeons have n boxes to enter. At least one box has more than one pigeon.

Ex If X and Y are sets and $|X| > |Y|$, then any function f from X to Y is not 1-1. For f is assigning elements from X to elements from Y . By the pigeonhole principle two elements must be assigned the same element in Y . Hence f is not 1-1.

If $|X| = |Y|$ and f is not onto then the elements in X are assigned to a smaller number of elements in Y . Hence two must be assigned

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the same element and f is not 1-1. Thus the contrapositive says if f is 1-1, then f is onto.

Suppose that f is onto, and $|X| = |Y|$. If f is not 1-1, two elements of X go to the same place in Y . Hence the image of f is not Y and f is not onto, a contradiction.

In summary, if $|X| = |Y|$ and $f: X \rightarrow Y$, then f is 1-1 if and only if f is onto.

Ex. Given m integers a_1, \dots, a_m , then there exist integers $0 \leq k < l \leq m$ such that $a_{k+1} + \dots + a_l$ is divisible by m .

Consider the m integers $a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_m$.

If any of these is divisible by m , then we are done ($k=0$ in this case).

If not, the m sums have remainders $\{1, \dots, m-1\}$ when divided by m

So two of the m remainder are

$$\text{equal: } a_{l+t} + a_k = bm + r_1,$$

$$a_{l+t} + a_0 = cm + r_2,$$

where $l > k$. Then, subtract,

$$a_{k+1} + a_l = (c-b)m \text{ and the}$$

result holds

Ex. A chess master has 11 weeks to prepare for a tournament. She will play one game every day (27 days) but no more than 12 games in any week (most played in the 11 weeks is 132). There exists a succession of days during which the chess master plays exactly 21 games.

Let $q_1 =$ number of games in day 1

$q_2 =$ " " " " day 1 + day 2

$q_3 =$ " " " " day 1 + day 2 + day 3

$q_t =$ " " " " in first t days

$$\leq 132$$

$$t \leq 77$$

Hence

$$1 \leq q_1 < q_2 < \dots < q_{77} \leq 132$$

Add 21 to each term

$$22 \leq q_1 + 21 < q_2 + 21 < \dots < q_{77} + 21 \leq 153$$

The 2 sequences have 154 members and take on up to 153 numbers. Two must be equal

The equal ones must come from the different sequences. Hence

There exist s and t such that

$$q_s = q_t + 21$$

Hence on day $t+1, t+2, \dots, s$, 21 games were played.

The Chinese Remainder Theorem is a useful number theory result which allows problems to be solved with smaller numbers and put together to give results

from Recall (all symbols are integers)

1. a and b are relatively prime if their only common positive divisor is 1

2. $a \equiv b \pmod{m}$ if m divides $a - b$. This is equivalent to $a = b + gm$ for some g .

C.R.T. Let m and n be relatively prime integers and $0 \leq a \leq m-1$, $0 \leq b \leq n-1$. Then there is a positive integer x such that $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$

Proof Consider

$$* \quad a, m+a, 2m+a, \dots, (n-1)m+a$$

When dividing by m , each of those integers has remainder a

Suppose 2 of them have the same remainder, r , when dividing by n

Let them be $lm+a$ and $jm+a$

$$0 \leq l < j \leq n-1$$

$$\text{Then } lm+a = q_l n + r$$

$$jm+a = q_j n + r$$

$$\text{Subtract, } (j-l)m = (q_j - q_l)n$$

So n divides $(j-l)m$. But

n and m are relatively prime, hence n divides $j-l$

But $0 \leq l < j \leq n-1$ and n can not divide $j-l$. This contradiction

gives that no two of $*$ have the same remainder when divided by n . So each of $0, 1, \dots, n-1$ occurs as a remainder. So b does ($0 \leq b \leq n-1$). Let p be the integer with $0 \leq p \leq n-1$ such that $x = pm+a$ has remainder b when

divided by n . Then, for some q ,

$$x = qn + b$$

Since $x = pm + a$, x has the properties

Example. From the integers $1, \dots, 200$

Choose 101 of them. There are two of them, a and b , such that either a divides b or b divides a .

Proof. Every integer a can be written as $a = 2^n r$ where r is odd.

There are 100 odd integers between 1 and 200, so two of our chosen integers have the same r . Write

$$b = 2^m r. \text{ If } n > m, \text{ then}$$

$$\frac{a}{b} = 2^{n-m} \text{ and } a = 2^{n-m} b, \text{ } b \text{ divides } a$$

$$\text{If } m > n, \text{ then } b = 2^{m-n} a.$$

Problems Chapter 3

3, 4, 5, 9, 10, 16, 18, 19

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More on the Chinese Remainder Theorem.

We look at the result with a different proof.

Thm. Let n_1, \dots, n_k be positive integers such that $\gcd(n_i, n_j) = 1$ for all $i \neq j$.
Let a_1, \dots, a_k be positive integers.
There exists x such that

$$x \equiv a_i \pmod{n_i} \text{ for each } i$$

Proof. Let $M = n_1 \dots n_k$ and

$$N_j = \frac{M}{n_j}. \text{ Then } N_j \text{ and } n_j \text{ are}$$

relatively prime so there

exists b_j such that $b_j N_j \equiv 1 \pmod{n_j}$

$$\text{Let } x = b_1 N_1 a_1 + \dots + b_k N_k a_k$$

$$x \pmod{n_j} = b_j N_j a_j \pmod{n_j} = a_j$$

Ex Find x such that

$$x \equiv a_1 \pmod{5} \quad n_1 = 5$$

$$x \equiv a_2 \pmod{7} \quad n_2 = 7$$

$$x \equiv a_3 \pmod{9} \quad n_3 = 9$$

$$M = 5 \cdot 7 \cdot 9 = 315$$

$$N_1 = \frac{M}{n_1} = 63$$

$$N_2 = \frac{M}{n_2} = 45$$

$$N_3 = \frac{M}{n_3} = 35$$

$$b_1 N_1 = b_1 \cdot 63 \pmod{5} = b_1 \cdot 3 \pmod{5}$$

is 1 when $b_1 = 2$

$$b_2 N_2 = b_2 \cdot 45 \pmod{7} = b_2 \cdot 3 \pmod{7}$$

is 1 when $b_2 = 5$

$$b_3 N_3 = b_3 \cdot 35 \pmod{9} = b_3 \cdot 8 \pmod{9}$$

is 1 when $b_3 = 8$

$$x = 2 \cdot 63 \cdot a_1 + 5 \cdot 45 \cdot a_2 + 8 \cdot 35 \cdot a_3$$

$$x \pmod{5} = a_1 \quad x \pmod{7} = a_2 \quad x \pmod{9} = a_3.$$

Problem 60 P. 67

Store has 6 kinds of bagels.

Choose 15 bagels

a. How many collections?

$$x_1 + \dots + x_6 = 15 \quad x_i \geq 0 \quad \binom{15+6-1}{15} = \binom{20}{15}$$

b. One of each kind

$$x_1 + \dots + x_6 = 15 \quad x_i \geq 1 \quad y_i = x_i - 1$$

$$(y_1 + 1) + \dots + (y_6 + 1) = 15 \quad y_i \geq 0 \quad \binom{9+6-1}{9} = \binom{14}{9}$$

$$y_1 + \dots + y_6 = 9 \quad y_i \geq 0$$

c. Probability of one of each kind at least

$$\frac{\binom{14}{9}}{\binom{20}{15}}$$

d. At least 3 sesame

$$x_1 + \dots + x_6 = 15 \quad x_1 \geq 3 \quad x_i \geq 0 \quad z = 2$$

$$y_i = x_i - 3$$

$$(y_1 + 3) + \dots + y_6 = 15$$

$$y_1 + \dots + y_6 = 12 \quad y_i \geq 0 \quad \binom{12+6-1}{12} = \binom{17}{12}$$

e. Prob at least 3 sesame

$$\frac{\binom{17}{12}}{\binom{20}{15}}$$

Problem 63 p. 68

4 Dice, different colors)

Roll all 4. Total ways 6^4

a. Total number of dots is 6

3 1's 1 3 4 ways

2 1's 2 2's $\binom{4}{2} = 6$ ways

$$Pr = \frac{4+6}{6^4}$$

b. At most 2 dice show one dot

None 5^4 1 $4 \cdot 5^3$ 2 $\binom{4}{2} 5^2$

$$Pr = \frac{5^4 + 4 \cdot 5^3 + \binom{4}{2} 5^2}{6^4}$$

c. $\frac{5^4}{6^4}$

d. $\frac{P(6, 4)}{6^4}$

e. $\frac{\binom{4}{2} 6 \cdot 5 + \binom{4}{3} 6 \cdot 5 + \binom{4}{1} 6 \cdot 5}{6^4}$