

MA 416

LESSON 8

More on The Pigeonhole Principle

Theorem. Let q_1, \dots, q_n be positive integers. Suppose that

$q_1 + \dots + q_n - n + 1$ objects are placed in n boxes. Then either

box 1 contains at least q_1 objects

box 2 contains at least q_2 objects

⋮

box n contains at least q_n objects

Proof Suppose, not. Then the sum of the ~~obj~~ number of objects in the boxes is $\leq (q_1 - 1) + (q_2 - 1) + \dots + (q_n - 1) = (q_1 + \dots + q_n) - n$. That is less than $(q_1 + \dots + q_n) - n + 1$, a contradiction.

Ex. let $q_1 = 4$ $q_2 = 5$ $q_3 = 7$ $n = 3$

suppose $4 + 5 + 7 - 3 + 1 = 14$ objects are placed in 3 boxes with

box 1 contains $4 - 1 = 3$ objects

box 2 contains $5 - 1 = 4$ "

box 3 contains $7 - 1 = 6$ "

This is only 13 objects; less than 14.

2. An important special case is when $g_1 = \dots = g_n = r$. Then if

$n(r-1) + 1 = nr - n + 1$ objects are placed in n boxes, then one of the boxes must have at least r objects

Proof. If every box has less than r objects, only $n(r-1) = nr - n$ objects are accounted for (at most), a contradiction

Ex Let $r = 4$ $n = 3$

$n(r-1) + 1 = 3(4-1) + 1 = 10$ objects

in 3 boxes one must contain at least $r = 4$ objects

Ex Given 52 integers. Show that there exist 2 of them whose sum or difference is divisible by 100. Consider the integers mod 100. If a and b are 2 of them

$$a = g_1 100 + r_1 \quad b = g_2 100 + r_2$$

100 divides $a-b$ ($r_1 - r_2$) if and only if 100 divides $r_1 - r_2$ ($r_1 + r_2$). Pick one of the integers, say a . Let b run through the other 51 integers. There are 51 sums $a+b \pmod{100}$ and 51 differences $a-b \pmod{100}$

So two must be equal. If

$$a-b = q_1 100 + r \quad a-c = q_2 100 + r, \text{ then}$$

$$c-b = (q_1 - q_2) 100 \text{ and } 100 \text{ divides}$$

$c-b$. Likewise for

$$a-b = q_1 100 + r \quad a+c = q_2 100 + r \text{ and}$$

$$a+b = q_1 100 + r \quad a+c = q_2 100 + r \text{ so}$$

the result holds

Another result is if the average of n non-negative integers is greater than $r-1$, then one of the integers is greater than or equal to r .

Proof. Suppose m_1, \dots, m_n are the integers. If the result fails, then each $m_i \leq r-1$. Then

$$m_1 + \dots + m_n \leq n(r-1) \text{ and}$$

$$\frac{m_1 + \dots + m_n}{n} \leq r-1 \quad \text{a contradiction}$$

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Ex Prove that from a group of n people, there are 2 that have the same number of acquaintances.

Every person knows from 0 to $n-1$ people.

If 2 people know no-one, they are the two people

If everyone knows someone, n people know from 1 to $n-1$ people. By the pigeonhole principle, 2 people know the same number

Finally, if exactly one person knows no-bodies, then $n-1$ people know from 1 to $n-2$ people. Again the pigeonhole principle gives the answer

Ex. A large disk is divided into 200 pie shaped regions, as is a small disk. In the large disk, 100 regions are red, 100 are blue. In the small disk, some regions are red, the rest are blue, but maybe not 100 of each. The disks are placed so that their centers coincide. Show that a rotation of the disks gives at least 100 regions where the colors are the same.

Focus on one section of the small disk. Turn the small disk 200 times and the section will coincide color wise with the large disk exactly 100 times. This happens for each section of the small disk for a total of $100 \cdot 200$ total matches. The average color matches per position is

$$\frac{100 \cdot 200}{200} = 100. \text{ By the pigeonhole}$$

principle, there is some position of the disks that have 100 matches, at least

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Ex. A basket of fruit must have at least 8 apples or at least 6 bananas or at least 9 oranges. How many pieces of fruit are guaranteed to meet the requirement.

$$\text{Ans } (8 + 6 + 9) - 3 + 1 = 21,$$

Ex. There are 100 people at a party. Each person knows an even number of people (not including themselves). Claim: at least 3 people know the same number of people.

Each person knows either 0, 2, 4, ..., 98 people.

Ⓐ If 2 people know nobody, then each person knows from 2 to 96 people.

Homeworks

CH 3: 8, 14, 15, 16, 18, 19

Problems (Ch 3)

4. Show that if $n+1$ integers are chosen from $\{1, \dots, 2n\}$, then there are always 2 that differ by 1

Consider n sets $\{1, 2\}, \{3, 4\}, \dots, \{2n-1, 2n\}$ of the $n+1$ choices must be in the same set by the pigeonhole principle. Those numbers differ by 1

9. 10 people ages 1 to 60. Show we can always find 2 groups of these people which have their ages added up to the same number and the groups have no common members

There are $2^{10} = 1024$ subsets

For any group, the sum of the ages is between 1 and 600. Since $1024 > 600$, 2 of the sums are the same. In any people are in both subsets, delete them and the sums are the same