

M A 416

Lesson 9

Relations

1 Relations

Def Let S be a set. A relation R is any subset of $S \times S$.

Ex. Let $S = \{a, b, c\}$. Possible R 's

$$R = \{(a, a), (b, c)\}$$

$$R = \{(a, b), (b, c), (c, a)\}$$

Note that we are considering

$S \times S$ as ordered pairs. Hence

(a, b) and (b, a) are different in S

Usually R is obtained from some important properties.

Ex. Let $S = \mathbb{Z} \times \mathbb{Z}$, where \mathbb{Z} denotes

the integers. Let R be defined

by $(a, b) \in R$ if $a < b$. Hence

$(2, 3)$, $(-2, -1)$ and $(0, 4) \in R$

Often $(a, b) \in R$ is written as

$a R b$

2 Ex. Let S be a set and

~~P~~' $P(S)$ be all subsets of S

Define R by $A R B$ if $A \subseteq B$

where $A, B \subseteq S$

Ex $S = \{a, b, c\}$

$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Then $\{a\} R \{a, b\}$ but $\{c\} \not R \{a, b\}$
where $\not R$ means not related

R usually has special properties

- ① R is reflexive if $x R x$ for all $x \in S$
- ② R is irreflexive if $x \not R x$ for all $x \in S$
- ③ R is symmetric if for all $x, y \in S$,
 $x R y$ implies $y R x$
- ④ R is antisymmetric if for all $x, y \in S, x \neq y$,
when $x R y$, then $y \not R x$; i.e., $x R y$ and
 $y R x$ implies $x = y$
- ⑤ R is transitive if for all $x, y, z \in S$, $x R y$ and $y R z$ implies
 $x R z$.

3 Ex Let $S = \{a, b, c\}$

$$R = \{(a, a), (a, b), (b, a), (b, b), (c, c)\}$$

is reflexive, symmetric and Transitive.

Ex Let $S = \{a, b, c\}$

$$R = \{(a, b), (b, c)\}$$

is neither reflexive nor irreflexive

is not symmetric, is antisymmetric
and is transitive.

Ex Let $S = \{a, b, c\}$

what does R look like if it
is both symmetric and antisymmetric?

Ex. Let S be the positive integers and

$$R = \{(a, b); a \text{ divides } b \text{ evenly}\} \quad (\text{Usually})$$

This is said a divides b without
saying evenly)

R is reflexive, antisymmetric and
Transitive

- 4 Ex. Let S be the integers and
 $R = \{(a, b) / a \leq b\}$
Then R is reflexive, antisymmetric
and transitive. What if we
replace $a \leq b$ by $a < b$.
- Ex. Let X be a set and $S = P(X)$,
the set of all subsets of X .
Let $R = \{(a, b) / a \subseteq b\}$
Then R is reflexive, antisymmetric
and transitive. What if \subseteq is
replaced by \subset .
- Ex. Let S be the integers, n an
integer. $R = \{(a, b) / n \text{ divides } a - b\}$
We write $a \equiv b \pmod{n}$ and say
 a is congruent to $b \pmod{n}$
 R is reflexive, symmetric and
transitive.

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\mathcal{R} is a partial order on S if
 \mathcal{R} is reflexive, antisymmetric and
transitive. \mathcal{R} is a strict partial
order on S if \mathcal{R} is irreflexive
antisymmetric and transitive

Ex. $<$ is a strict partial order on \mathbb{Z} ,
the integers

\leq is a partial order on \mathbb{Z}

C is a strict partial order on $P(S)$

\subseteq is \subset partial order on $P(S)$

$|$, divides is a partial order on \mathbb{Z}

A set S with partial order \mathcal{R} is
called a partially ordered set,
or POSET.

For a relation \mathcal{R} on S , $x, y \in S$, then
 x and y are compatible if either
 $x \mathcal{R} y$ or $y \mathcal{R} x$

3 and 5 are not compatible when
 \mathcal{R} = divides and $S = \mathbb{Z}$

A partial order is a total order if
every pair of elements are compatible

Ex. Let $S = \text{positive integers}$
 \leq is a Total order on S
 $|$ is not a Total order on S

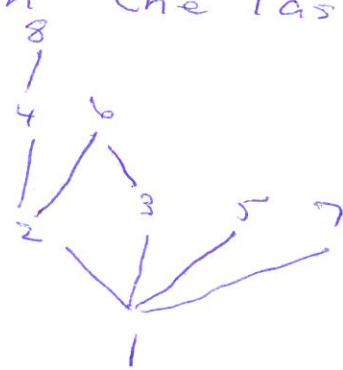
A finite set of elements can always
be linearly ordered in many ways

There is a 1-1 correspondence between
the total orders on a finite set
 S and the permutations of S .
To form a total order simply
list the elements ~~in~~ in a row
and write aRb if a is to the
left of b . This is a total order.
There are $n!$ ways of doing this,
each corresponds to a permutation
of the original way.

Let (S, R) be a poset and $a, b \in S$
 b is said to cover a if aRb
and there is no element c such that
 aRc and cRb . In a totally ordered
set, in the ordering each element
 b that follows a has aRb
↑
Immediately

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Ex Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $R = \text{l, divides}$
 Then 2 covers 1, 4 covers 2, 8 covers 4
 6 covers 3. The idea can be
 represented geometrically by a Hasse
 diagram. Plot the points putting
 x below y if xRy . The Hasse diagram
 for S in the last example is

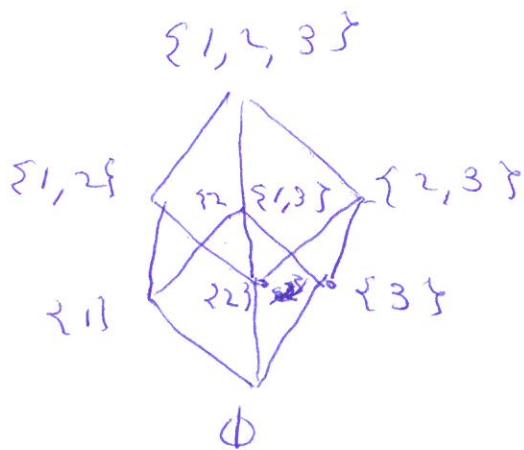


If the order is a total order then
 the Hasse diagram is a line

Ex $S = \{1, 2, \dots, 8\}$ and R is $<$



$$\text{Ex} \quad X = \{1, 2, 3\} \quad S = P(X) \quad R = \subseteq$$



Let R_1 and R_2 be two partial orders on S . Then R_2 is an extension of R_1 , if whenever $aR_1 b$ then $aR_2 b$. Every partial order has an extension that is a linear order; a total order.

The way to show this gives the way to construct the linear order

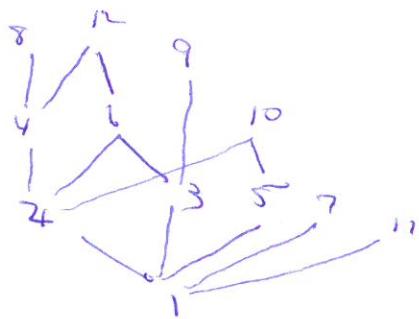
Choose an element $x_1 \in S$ such that

there is no $x \in S$ with $x R x_1$.

Delete x_1 and repeat the process.

This gives the linear order if we define R_1 by letting $a R_1 b$ if a was picked before b .

Ex $S = \{1, \dots, 12\}$ $R_1 = |$



is the cover relation for R_1 .

Pick $1, 2, 3, 5, 7, 11, 4, 6, 10, 8, 9, 12$

and let R_2 be $a R_2 b$ if

a is to the left of b . The relation R_2 is a total order and an extension of R_1 . There are other total orders that are an extension of R_1 ,

$1, 4, 3, 2, 6, 8, 12, 9, 5, 10, 7, 11$

for instance

Another very important class of relations are equivalence relations, usually denoted by \sim . They are relations that are reflexive, symmetric and transitive.

Equals is an equivalence relation for \mathbb{Z} and for $P(X)$. Congruence on \mathbb{Z} is an equivalence relation: Fix n .

aRb if $a-b \equiv n \pmod{k}$ for some integer k . Show this satisfies the properties for equivalence. Usually this is written as $a \equiv b \pmod{n}$ when aRb .

Equivalence is used to partition the set S :

Def Let S be a set. Subsets A_1, A_k are a partition of S if $A_i \cap A_j = \emptyset$ for $i \neq j$ and $S = A_1 \cup \dots \cup A_k$

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If R is an equivalence relation, let

$A_a = \{x \mid aRx\}$. $a \in A$, so S is the union of its equivalence classes, A_a .

We need to show $A_a \cap A_b = \emptyset$ if

$A_a \neq A_b$. Let $x \in A_a \cap A_b$. Then

aRb , bRx . Then aRx , $xRb \rightarrow aRb$ and bRx , aRx . Then aRb and bRa and aRa . So $b \in A_a$. If $y \in A_b$ then bRy and $y \in A_b$. If $y \in A_a$ then aRy and $y \in A_b$. So $A_b \subseteq A_a$. Likewise $A_a \subseteq A_b$ and $A_a = A_b$.

Ex $n=5$ for congruence. Then

$$a = ng + r, \quad 0 \leq r \leq 4 \quad S.$$

$A_0, A_1, A_2, A_3, A_4, A_5$ are the equivalence classes where $A_r = \{a \mid a \equiv ng + r\}$

i.e; those a whose remainder is r when a is divided by n .

Problems P. 122

$$36, 37, 42, 43, 45, 46, 48$$