

MA 416

Lesson 14

Chains and Antichains

Chains and Antichains in a general partially ordered set.

Let (S, \leq) be a finite poset. An antichain is a subset A of S such that if $a, b \in A$, $a \neq b$ then $a \not\leq b$ and $b \not\leq a$. A chain is a subset of ~~$\subseteq S$~~ such that either $a \leq b$ or $b \leq a$ for all $a, b \in S$.

Ex $S = \{1, 2, \dots, 10\}$ and \leq is divides. An antichain is $\mathcal{Q} = \{2, 3, 5, 7\}$ and a chain is $\mathcal{C} = \{2, 4, 8\}$. Note that $\mathcal{Q} \cap \mathcal{C}$ is either empty or contains just one element. For if $a, b \in \mathcal{Q} \cap \mathcal{C}$, then $a \leq b$ or $b \leq a$ because of \mathcal{C} and $a \neq b$ and $b \neq a$ because of \mathcal{Q} , a contradiction.

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Example. Let $S = \{1, 3, \dots, 10\}$ and \leq is divides. $A = \{4, 6, 7, 9, 10\}$ is an antichain of size 5. $C = \{1, 2, 4, 8\}$ is a chain of size 4. We consider partitions of S .

into chains and also into antichains.

Suppose there is a chain C of size r .

Any antichain intersects C in at most one point. If a partition of S into antichains has less than r elements, then some member of the partition intersects C in more than one place, a contradiction. Hence there are at least r elements in the antichain. Using a similar argument, if A is an antichain of size s , then there must be at least s chains in a partition of S .

Ex $S = \{1, \dots, 10\}$

$$A = \{4, 6, 7, 9, 10\} \quad |A| = 5$$

Chain partition $\{1, 2, 4, 8\}, \{3, 6\}, \{5, 10, 7\}, \{9\}$

while $C = \{1, 2, 4, 8\}$

Antichain partition $\{6, 7, 8, 9, 10\}, \{3, 4, 5\}, \{2\}, \{1\}$

Def. Let S be a POSET. A minimal element in S is one with the property that there exist no element x with $x \leq a$ ($x \neq a$). A maximal element in S is one such that there exist no yes with $b \leq y$ ($y \neq b$)

In the previous example, 1 is the only minimal element while $\{6, 7, 8, 9, 10\}$ are all maximal elements.

The set of all minimal elements, maximal elements, form antichains.

Theorem A. Let (S, \leq) be a POSET (finite) and let r be the size of a largest chain in S . Then S can be partitioned into r but no fewer antichains.

Theorem B. Let (S, \leq) be a POSET and let s be the size of a largest antichain in S . Then S can be partitioned into s but not fewer chains.

4 Ex $S = \{1, 2, \dots, 10\}$

Chain $A = \{1, 2, 4, 8\}$

Antichains $\{6, 7, 8, 9, 10\} \quad \{3, 4, 5\} \quad \{2\} \quad \{1\}$

Antichain $\{4, 7, 8, 9, 10\}$

Chains $\{1, 2, 4, 8\} \quad \{3, 6\}, \{5, 10\} \quad \{7\} \quad \{9\}$

Proof of Theorem BA Let r be the size of the largest chain in S . We know it takes at least ~~r~~ antichains of size r to partition S . We need to show there is an antichain of size r that partitions S .

Let $S_1 = S$ and let $A_1 = \{\text{minimal elements in } S_1\}$. Delete them

To get S_2 let A_2 be the minimal elements of S_2 . Delete the minimal elements from A_2

They are all ~~to~~ to get S_2

Each of A_1, A_2 are antichains and this process continues for r steps to partition S into r antichains

Ex. $S = \{1, 2, \dots, 10\}$

Chain $C = \{1, 2, 4, 8\}$

$$A_1 = \{1\} \quad S_1 = \{2, 3, \dots, 10\}$$

$$A_2 = \{2, 3, 5, 7\} \quad S_2 = \{4, 6, 8, 10\}$$

$$A_3 = \{4, 6, 10\} \quad S_3 = \{8\}$$

$$A_4 = \{8\}$$

Notice that for each step we get one element of the chain so the number of antichains = number of elements in the chain. Also notice that each A_i is an antichain since they are the minimal elements of poset S_{i-1} .

Ex. Let X be a set, $S = P(X)$ the poset of subsets of X . A maximal chain: $\emptyset \subseteq \{1\} \subseteq \{1, 2\} \subseteq \dots \subseteq \{1, 2, \dots, n\}, n+1$ elements in the chain. The antichains are all elements of a given size, size $0, 1, \dots, n$. So there are $n+1$ elements in the partition of S into antichains.

Theorem B. $|S| = n$ and largest
antichain has m elements. We
need to show there is a partitions
of S into m chains. Use induction
on $n = |S|$. If $n=1$, the result
is trivial.

Assume $n > 1$.

Case 1. There is an antichain, that is neither the set of all minimal elements, nor the set of all maximal elements and A has m elements.

Let $A^+ = \{x \in S \text{ with } a \leq x \text{ for some } a \in A\}$,

and $A^- = \{x \in S \text{ with } x \leq a \text{ for some } a \in A\}$

A^+ is all elements above A and A^- is all elements below A . Then

$A^+ \neq S$ ($|A^+| < |S|$) since there

is a minimal element not in A

$A^- \neq S$ ($|A^-| < |S|$) since there

is a maximal element not in A .

$A^+ \cap A^- = A$ for if $x \in A^+ \cap A^-$,

$x \notin A$, then $a_1 < x < a_2$ for $a_1, a_2 \in A$.

This contradicts that A is an antichain.

$A^+ \cup A^- = S$ since if $x \notin A^+ \cup A^- \rightarrow A \cup \{x\}$ would be an antichain of larger size than A

Use induction on $A^+ \cup A^-$. Then A^+ can be partitioned into m chains E_1, \dots, E_m and A^- can be partitioned into m chains of A are also minimal elements of A^+ and so the first element of chains E_1, \dots, E_m , give the chains together in pairs to form m chains that partition S .

Case 2. There are at most 2 antichains of size m and those are the set of all max elements and the set of all min elements. Let x be a minimal element

and y be a maximal element, so

$x \leq y$. The size of a largest antichain of $S - \{x, y\}$ is $m-1$. By induction $S - \{x, y\}$ can be partitioned into $m-1$ chains. Those chains along with chain $\{x, y\} \setminus x \leq y$ give a partition of S into m chains.

Problems chapter 5:

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