

mA 416

Lesson 17

Review: Let S be the multiset

$$S = \{m_1 q_1, \dots, m_k q_k\} \text{ where}$$

the m_i are non negative integers or ∞ . How many r -combinations are there? If there are $x_1 q_1$'s,

$x_2 q_2$'s ... $x_k q_k$'s, then

$$x_1 + \dots + x_k = r \quad x_i \geq 0$$

Ans, we saw, is $\binom{r+k-1}{k-1}$

We also considered the case

when also $x_i \geq s_i$ some s_i , $i=1, \dots, k$

$$\text{so } x_1 + \dots + x_k = r \quad x_i \geq s_i$$

$$\text{we let } y_i = x_i - s_i$$

$$x_i = y_i + s_i$$

$$(y_1 + s_1) + \dots + (y_k + s_k) = r \quad y_i \geq 0$$

$$y_1 + \dots + y_k = r - (s_1 + \dots + s_k)$$

$$\text{Ans } \binom{k-1 + (r - (s_1 + \dots + s_k))}{k-1}$$

In all these cases, $m_j \geq r$ for each j
(often $m_j = \infty$)

Next we consider the case when

$$m_j \leq \text{some } P_j \quad j=1, \dots, k$$

This case is done by inclusion-exclusion, using the results we just recalled

Ex Find the number of solutions
to $x_1 + x_2 + x_3 = 20$ where

$$x_1 \leq 3 \quad x_2 \leq 2 \quad x_3 \leq 5$$

Let P_1 = solutions when $x_1 \geq 4$

P_2 = solutions when $x_2 \geq 3$

P_3 = solutions when $x_3 \geq 6$

Then $|\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3|$ is the number
of solutions to $x_1 + x_2 + x_3 = 20$,
 $x_1 \leq 3, x_2 \leq 2, x_3 \leq 5$

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We use

$$\begin{aligned} |\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3| &= (S) - [|P_1| + |P_2| + |P_3|] \\ &+ [|P_1 \cap P_2| + |P_1 \cap P_3| + |P_2 \cap P_3|] \\ &- |P_1 \cap P_2 \cap P_3| \end{aligned}$$

 $S = \text{all solutions:}$

$$x_1 + x_2 + x_3 = 20 \quad x_i \geq 0$$

$$\text{Ans: } \binom{20+3-1}{3-1} = \binom{22}{2}$$

$$P_1 \quad x_1 + x_2 + x_3 = 20 \quad x_1 \geq 4$$

$$\text{Let } y_1 = x_1 - 4 \quad y_2 = x_2 \quad y_3 = x_3$$

$$y_1 + 4 + y_2 + y_3 = 20$$

$$y_1 + y_2 + y_3 = 16 \quad y_i \geq 0$$

$$\binom{16+3-1}{3-1} = \binom{18}{2}$$

$$P_2 \quad x_1 + x_2 + x_3 = 20 \quad x_2 \geq 3 \quad \text{becomes}$$

$$y_1 + (y_2 + 3) + y_3 = 20$$

$$y_1 + y_2 + y_3 = 17$$

$$\binom{17+3-1}{3-1} = \binom{19}{2}$$

$$P_3 \quad x_1 + x_2 + x_3 = 20 \quad x_3 \geq 6$$

$$y_1 + y_2 + y_3 + 6 = 20$$

$$y_1 + y_2 + y_3 = 14 \quad y_1 \geq 0$$

$$\binom{14+3-1}{2} = \binom{16}{2}$$

$$P_1 \cap P_2 \quad x_1 + x_2 + x_3 = 20 \quad x_1 \geq 4 \quad x_3 \geq 3$$

$$(y_1 + 4) + (y_2 + 3) + y_3 = 20$$

$$y_1 + y_2 + y_3 = 13 \quad y_1 \geq 0$$

$$\binom{13+3-1}{3-1} = \binom{15}{2}$$

$$x_1 \geq 4 \quad x_3 \geq 6$$

$$P_1 \cap P_3$$

$$(y_1 + 4) + (y_2) + (y_3 + 6) = 20$$

$$y_1 + y_2 + y_3 = 10$$

$$\binom{10+3-1}{3-1} = \binom{12}{2}$$

$$P_2 \cap P_3 \quad x_1 + x_2 + x_3 = 12 \quad x_2 \geq 3 \quad x_3 \geq 1$$

$$(y_1 + 3) + (y_2) + (y_3 + 6) = 20$$

$$y_1 + y_2 + y_3 = 11 \quad y_2 \geq 0$$

$$\binom{11+3-1}{3-1} = \binom{13}{2}$$

$$P_1 \cap P_2 \cap P_3$$

$$x_1 \geq 4 \quad x_2 \geq 3 \quad x_3 \geq 6$$

$$(y_1 + 4) + (y_2 + 3) + (y_3 + 6) = 20$$

$$y_1 + y_2 + y_3 = 7$$

$$\binom{7+3-1}{3-1} = \binom{9}{2}$$

So

$$|\tilde{P}_1 \cap \tilde{P}_2 \cap \tilde{P}_3| = \binom{22}{2} - \left[\binom{18}{2} + \binom{19}{2} + \binom{15}{2} \right] \\ + \left[\binom{15}{2} + \binom{12}{2} + \binom{13}{2} \right] - \binom{9}{2}$$

Ex Find the number of boxes of donuts that can be made given we have 3 glazed, 4 lemon filled and 5 apple cinnamon that can be used.

We use inclusion-exclusion

$|S|$ is the number boxes with any combination of the three types

x_1 = number of glazed

x_2 = number of lemon

x_3 = number of cinnamon boxes

$$x_1 + x_2 + x_3 = 12 \quad x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

$$\text{Ans} \left(\begin{matrix} 12+3-1 \\ 3-1 \end{matrix} \right) = \left(\begin{matrix} 14 \\ 2 \end{matrix} \right)$$

P₁ more than 3 glazed $x_1 \geq 4$

$$x_2 \geq 0 \quad x_3 \geq 0 \quad x_1 + x_2 + x_3 = 12$$

$$y_1 = x_1 - 4 \quad y_2 = x_2 \quad y_3 = x_3$$

$$(y_1 + 4) + y_2 + y_3 = 12$$

$$y_1 + y_2 + y_3 = 8 \quad y_i \geq 0$$

$$\left(\begin{matrix} 8+3-1 \\ 3-1 \end{matrix} \right) = \left(\begin{matrix} 10 \\ 2 \end{matrix} \right)$$

P₂ more than 4 lemon

$$y_2 = x_2 - 4$$

$$y_1 = x_1$$

$$y_3 = x_3$$

$$y_1 + (y_2 + 4) + y_3 = 12 \quad y_i \geq 0$$

$$y_1 + y_2 + y_3 = 8$$

$$\left(\begin{matrix} 10 \\ 2 \end{matrix} \right)$$

$$\sim 11$$

$$\left(\begin{matrix} 8+3-1 \\ 3-1 \end{matrix} \right)$$

? P_2 more than 4 lemon

$$x_2 \geq 5$$

$$x_1 + x_2 + x_3 = 12$$

$$y_1 + (y_2 + 5) + y_3 = 12 \quad y_1 + y_2 + y_3 = 7$$

$$\binom{7+3-1}{2} = \binom{9}{2}$$

P_3 more than 5 apple spice $x_3 \geq 6$

$$y_1 + y_2 + (y_3 + 6) = 12$$

$$y_1 + y_2 + y_3 = 6 \quad y_1 \geq 0 \quad \binom{6+3-1}{3-1} = \binom{8}{2}$$

$P_1 \cap P_2 \quad x_1 \geq 4 \quad x_2 \geq 5$

$$x_1 + x_2 + x_3 = 12 \quad y_1 = x_1 - 4 \quad y_2 = x_2 - 5$$

$$(y_1 + 4) + (y_2 + 5) + y_3 = 12$$

$$y_1 + y_2 + y_3 = 3$$

$$\binom{3+3-1}{3-1} = \binom{5}{2}$$

$P_1 \cap P_3 \quad x_1 \geq 4 \quad x_3 \geq 6$

$$(y_1 + 4) + y_2 + (y_3 + 6) = 12$$

$$y_1 + y_2 + y_3 = 2$$

$$\binom{2+3-1}{2} = \binom{4}{2}$$

$P_2 \cap P_3 \quad x_2 \geq 5 \quad x_3 \geq 6$

$$y_1 + y_2 + y_3 = 1$$

$$\binom{1+3-1}{3-1} = \binom{3}{2}$$

$$P_1 \cap P_2 \cap P_3 = \quad x_1 \geq 4 \quad x_2 \geq 8 \quad x_3 \geq 6$$

$$(y_1 + 4) + (y_2 + 5) + (y_3 + 6) = 12$$

$$y_1 + y_2 + y_3 = -3$$

$$PA_{ns} = 6$$

S_0

$$\begin{aligned} |\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3| &= \binom{14}{2} - [\binom{10}{2} + \binom{9}{2} + \binom{18}{2}] \\ &+ [\binom{5}{2} + \binom{4}{2} + \binom{3}{2}] = 0 \end{aligned}$$

Ex Find all integer solutions to

$$x_1 + x_2 + x_3 = 20$$

where $1 \leq x_1 \leq 3$

$$1 \leq x_2 \leq 4$$

$$2 \leq x_3 \leq 5$$

First, reduce this to having variables less greater than or equal to 0.

$$y_1 = x_1 - 1$$

$$0 \leq y_1 \leq 2$$

$$y_2 = x_2 - 1$$

$$0 \leq y_2 \leq 3$$

$$y_3 = x_3 - 2$$

$$0 \leq y_3 \leq 3$$

$$(y_1 + 1) + (y_2 + 1) + (y_3 + 1) = 20$$

$$y_1 + y_2 + y_3 = 16$$

$$0 \leq y_1 \leq 2 \quad 0 \leq y_2 \leq 3 \quad 0 \leq y_3 \leq 3$$

$$P_1 \quad y_1 \geq 3$$

$$y_2 \geq 4$$

$$y_3 \geq 4$$

$$\begin{aligned} |\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3| &= |S| - (|P_1| + |P_2| + |P_3|) \\ &\quad + (|P_1 \cap P_2| + |P_1 \cap P_3| + |P_2 \cap P_3|) \\ &\quad - |P_1 \cap P_2 \cap P_3| \end{aligned}$$

$$\text{P}_5 \quad y_1 + y_2 + y_3 = 11 \quad \begin{pmatrix} 16+3-1 \\ 2 \end{pmatrix}$$

$$y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0$$

$$P_1 \quad y_1 \geq 3$$

$$(z_1 + 3) + z_2 + z_3 = 16 \quad \begin{pmatrix} 13+3-1 \\ 2 \end{pmatrix}$$

$$z_1 + z_2 + z_3 = 13$$

$$P_2 \quad y_2 \geq 4$$

$$z_1 + (z_2 + 4) + z_3 = 16 \quad \begin{pmatrix} 12+3-1 \\ 2 \end{pmatrix}$$

$$z_1 + z_2 + z_3 = 12$$

$$P_3 \quad y_3 \geq 4$$

$$z_1 + z_2 + (z_3 + 4) = 16$$

$$z_1 + z_2 + z_3 = 12$$

$$P_1 \cap P_2 \quad (z_1 + 3) + (z_2 + 4) + z_3 = 16 \quad \begin{pmatrix} 9+3-1 \\ 2 \end{pmatrix}$$

$$z_1 + z_2 + z_3 = 9$$

$$P_1 \cap P_3 \quad (z_1 + 3) + z_2 + (z_3 + 4) = 16 \quad \begin{pmatrix} 9+3-1 \\ 2 \end{pmatrix}$$

$$z_1 + z_2 + z_3 = 9$$

$$P_2 \cap P_3 \quad (z_1 + (z_2 + 4)) + (z_3 + 4) = 16 \quad \begin{pmatrix} 8+3-1 \\ 2 \end{pmatrix}$$

$$z_1 + z_2 + z_3 = 8$$

$$P_1 \cap P_2 \cap P_3 \quad (z_1 + 3) + (z_2 + 4) + (z_3 + 4) = 16 \quad \begin{pmatrix} 5+3-1 \\ 2 \end{pmatrix}$$

$$z_1 + z_2 + z_3 = 5$$

$$|\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3| = \binom{18}{2} - [\binom{15}{2} + \binom{14}{2} + \binom{12}{2}] - \left[\binom{11}{2} + \binom{10}{2} + \binom{8}{2} \right]$$

In each calculation we get to

$$z_1 + z_2 + t_3 = c$$

$$t_1 \geq 0, t_2 \geq 0, t_3 \geq 0$$

It is with these inequalities
that we can pass to the
binomial coefficient $\binom{c+3-1}{3-1}$

Note that in P, we have

$y_1 \leq 2$. We change for the
complement to $y_1 \geq 3$

If we had $y_1 < 2$, then the
complement would use $y_1 \geq 2$

Problems Chapter 6

5, 6, 7, 9

Problem 3 page 198

Find the number of integers between 1 and 10 000 that are not perfect squares and not perfect cubes.

A_1 = perfect squares

A_2 = perfect cubes

$S = \{1, \dots, 10000\}$

$$|S| = 10000$$

$A_1 : 10^2 = 10000$, There are 100 perfect squares

$$A_2 : 21^3 = 9261 \quad |A_2| = 21$$

$$22^3 = 10648$$

$A_1 \cap A_2$ perfect cube and perfect square

$$1^6 = 1 = 1^3 = 1^2 \quad 2^6 = 4^3 = 8^2 = 64$$

$$3^6 = 16^3 = 64^2 > 10000$$

5⁶ is too big

$$|A_1 \cap A_2| = 4$$

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2| &= |S| - (|A_1| + |A_2|) + |A_1 \cap A_2| \\ &= 10000 - (100 + 21) + 4 \\ &= 9883 \end{aligned}$$