

M A 416

Lesson 18

① Start with an equation example  
using inclusion-exclusion

Find the number of integral solutions to

$$x_1 + x_2 + x_3 + x_4 = 18$$

$$1 \leq x_1 \leq 5 \quad -2 \leq x_2 \leq 4 \quad 0 \leq x_3 \leq 5 \quad 3 \leq x_4 \leq 9$$

To handle the right hand inequalities we use the technique from section 2.5

$$\text{Let } y_1 = x_1 - 1 \quad y_2 = x_2 + 2$$

$$y_3 = x_3 - 0 \quad y_4 = x_4 - 3$$

Substitute into given equation

$$(y_1 + 1) + (y_2 - 2) + (y_3 + 0) + (y_4 + 3) = 18$$

$$y_1 + y_2 + y_3 + y_4 = 16$$

$$0 \leq y_1 \leq 4 \quad 0 \leq y_2 \leq 6 \quad 0 \leq y_3 \leq 5 \quad 0 \leq y_4 \leq 6$$

Now we use inclusion exclusion on these  
 $S =$  all solutions (all non negative solutions)

$$y_1 + y_2 + y_3 + y_4 = 16$$

$$0 \leq y_1 \leq 4 \quad 0 \leq y_2 \leq 6 \quad 0 \leq y_3 \leq 5 \quad 0 \leq y_4 \leq 6$$

$$|S| = \binom{16+4-1}{4-1} = \binom{19}{3} = 969$$

$$\text{Let } P_1: y_1 \geq 5 \quad P_2: y_2 \geq 7 \\ P_3: y_3 \geq 6 \quad P_4: y_4 \geq 7$$

$$\text{We want } |P_1 \cap P_2 \cap P_3 \cap P_4|$$

$|S| = 969$  from last page

$$P_1: y_1 + y_2 + y_3 + y_4 = 16$$

$$y_1 \geq 5 \quad y_2, y_3, y_4 \geq 0$$

$$z_1 = y_1 - 5 \quad z_2 = y_2 \quad z_3 = y_3 \quad z_4 = y_4$$

$$z_1 + 5 + z_2 + z_3 + z_4 = 16$$

$$z_1 + z_2 + z_3 + z_4 = 11$$

$$|P_1| = \binom{11+4-1}{4-1} = \binom{14}{3} = 364$$

$$P_2: y_1 + y_2 + y_3 + y_4 = 16 \quad y_2 \geq 7$$

$$z_1 + (z_2 + 7) + z_3 + z_4 = 16$$

$$z_1 + z_2 + z_3 + z_4 = 9$$

$$\binom{9+4-1}{4-1} = \binom{12}{3}$$

$$P_3: y_3 \geq 6$$

$$z_1 + z_2 + (z_3 + 6) + z_4 = 16$$

$$z_1 + z_2 + z_3 + z_4 = 10$$

$$\binom{10+4-1}{3} = \binom{13}{3}$$

$$P_4: y_4 \geq 7$$

$$z_1 + z_2 + z_3 + (z_4 + 7) = 16$$

$$z_1 + z_2 + z_3 + z_4 = 9$$

$$\binom{9+4-1}{3} = \binom{12}{3}$$

3

Now the intersection of  $P_1$  and  $P_2$

$$|P_1 \cap P_2| \quad y_1 \geq 5 \quad y_2 \geq 7 \quad y_3 \geq 0 \quad y_4 \geq 0$$

$$(z_1 + 5) + (z_2 + 7) + z_3 + z_4 = 16$$

$$z_1 + z_2 + z_3 + z_4 = 4 \quad z_i \geq 0$$

$$\text{Get } \binom{4+4-1}{4-1} = \binom{7}{4}$$

$|P_1 \cap P_3|$  with  $y_1 \geq 5 \quad y_2 \geq 0 \quad y_3 \geq 6 \quad y_4 \geq 0$   
is done the same, giving  $\binom{8}{3}$

$$|P_1 \cap P_3|: \binom{6}{3} \quad |P_1 \cap P_4|: \binom{7}{4} \quad |P_2 \cap P_3|: \binom{5}{2}$$

$$|P_3 \cap P_4|: \binom{6}{3}$$

$$|P_1 \cap P_2 \cap P_3| \quad (z_1 + 5) + (z_2 + 7) + (z_3 + 6) + z_4 = 16$$

$$\rightarrow z_1 + z_2 + z_3 + z_4 = -2 \quad \text{No solutions}$$

The rest  $P_1 \cap P_2 \cap P_4$ ,  $P_2 \cap P_3 \cap P_4$ ,  $P_1 \cap P_3 \cap P_4$   
and  $|P_1 \cap P_2 \cap P_3 \cap P_4|$  all give 0

$$|\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3 \cap \bar{P}_4| = |S| - [(P_1 \cup P_2 \cup P_3 \cup P_4) \cap S]$$

$$+ [P_1 \cap P_2 + P_1 \cap P_3 + P_1 \cap P_4 + P_2 \cap P_3]$$

$$+ [P_2 \cap P_4 + P_3 \cap P_4] - |P_1 \cap P_2 \cap P_3 \cap P_4|$$

Substituting every calculation  
in gives 55

## DERANGEMENTS

Given  $\{1, \dots, n\}$  Find all permutations such that no integer is in its beginning position. These permutations are called derangements.

We derive a formula for the number of derangements using inclusion-exclusion.

Let  $P_i$  be the permutations with  $i$  in the  $i$  position.

We are looking for  $|\bar{P}_1 \cap \dots \cap \bar{P}_n|$

This is

$$|\bar{P}_1 \cap \dots \cap \bar{P}_n| = |S| - \binom{n}{1} (\text{all permutations of } n-1 \text{ elements}) + \binom{n}{2} (\text{all permutations of } n-2 \text{ elements}) - \binom{n}{3} (\text{all permutations of } n-3 \text{ elements}) + (-1)^n \binom{n}{n} (\text{all permutations of } 1 \text{ element}) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \dots + \binom{n}{n} 1!$$

$$= n! - n! + \frac{n!}{2!} - \frac{n!}{3!} + \dots + (-1)^n \frac{n!}{n!}$$

$$= n! \left[ 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right] = D_n$$

$D_n$  stands for the number of derangements  
of  $\{1, \dots, n\}$

$$n=2 \quad \left( \begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix} \right) \quad D_2 = 1$$

$$n=3 \quad \left( \begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix} \right) \quad \left( \begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix} \right) \quad D_3 = 2$$

$$D_4 = 4! \left[ 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} \right] =$$

$$4! \left( 1 - \frac{4-1}{24} \right) = 9$$

$$D_5 = 5! \left( 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} \right)$$

$$= 5! \frac{60-20+5-1}{120} = 44$$

$$\text{Check that } D_6 = 265, \quad D_7 = 1854$$

Ex Find the number of permutations

where exactly one element is fixed

The are  $n$  choices for the

fixed element. Then, the remaining elements

are a derangement of  $n-1$ ,  $D_{n-1}$  in total

Answer =  $n D_{n-1}$

Ex. Find the number of permutations where exactly 2 elements are fixed.

There are  $\binom{n}{2}$  choices for the two fixed elements and the remaining  $n-2$  have none fixed,  $D_{n-2}$  of them.

$$\text{Total} = \binom{n}{2} D_{n-2}$$

Ex. Find the number of permutations with at most two fixed elements.

$$\text{Exactly none} + \text{Exactly one} + \text{Exactly two}$$

$$= D_n + n D_{n-1} + \binom{n}{2} D_{n-2}$$

We state the evaluation of  $D_n$  as

Theorem.  $D_n = n! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^{\frac{n}{2}} \frac{1}{\frac{n}{2}!} \right]$

Recall that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^{n+1} \frac{1}{(n+1)!} + \dots$$

$$= \frac{D_n}{n!} + (-1)^{n+1} \frac{1}{(n+1)!} + \dots$$

In this alternating series, the terms from  $(-1)^{n+1} \frac{1}{(n+1)!}$  on contribute a total

$$\leq \frac{1}{(n+1)!} \quad \text{So } |e^{-1} - \frac{D_n}{n!}| < \frac{1}{(n+1)!}$$

So  $e^{-1}$  is almost the same as

$\frac{D_n}{n!}$  when  $n$  is not too large

and can be used to find  $\frac{D_n}{n!}$

Notice that  $\frac{D_n}{n!}$  is almost the

same for each  $n$ ,  $e^{-1}$ . So

for practical purposes

$$\frac{D_{10}}{10!} = \frac{D_{100}}{100!} = \frac{D_{1000}}{1000!}$$

This quantity stands for the probability of picking a permutation and getting a derangement =  $e^{-1}$ .

Theorem  $D_n = (n-1)(D_{n-2} + D_{n-1})$

when  $n = 3, 4, 5, \dots$

Since  $D_1 = 0$   $D_2 = 1$

$$D_3 = (3-1)(0+1) = 2$$

$$D_4 = (4-1)(1+2) = 3 \cdot 3 = 9$$

$$D_5 = (5-1)(2+9) = 44$$

$$D_6 = (6-1)(9+44) = 265.$$

Proof of Theorem

Consider derangements of  $\{1, \dots, n\}$

One of  $2, \dots, n$  is in the first position

No matter which, ~~derangement~~ the same number of derangements follow the number in position 1. Hence

$$D_n = (n-1) d_n \quad d_n = \text{number}$$

of derangements with 2 in the first position. They look like

$$2 \ l_2 \ l_3 \dots \ l_n$$

$$l_2 \neq 2 \quad l_3 \neq 3 \quad l_n \neq n$$

~~skip~~ 10  
Partition  $d_n$  further according to  
 $d_2 = 1$  or  $d_2 \neq 1$

$d_n'$  = number derangements of the form

$$2 \cdot 1 \cdot l_3 \cdots l_n$$

$$d_n'' : 2 l_2 \cdots l_n \quad l_n \neq 1 \quad l_3 \neq 3, \dots, l_n \neq n$$

Then  $d_n = d_n' + d_n'' \rightarrow$

$$D_n = (n-1) d_n = (n-1)(d_n' + d_n'')$$

$d_n'$  is the number of derangements

of  $\{3, 4, \dots, n\}$ , so  $d_n' = D_{n-2}$

$d_n''$  is the number der-pertnuTations  
of  $\{1, 3, 4, \dots, n\}$  where 1 is not in first  
position 3 is not in second position ...

and n is not in n<sup>th</sup> position. So

$d_n'' = D_{n-1}$ . Summarizing

$$D_n = (n-1)(d_n' + d_n'') = (n-1)(D_{n-2} + D_{n-1})$$

which is the result

The last equation can be written as

$$D_n - n D_{n-1} = -[D_{n-1} - (n-1) D_{n-2}]$$

This recursive expression can be continued

$$= (-1)^2 [D_{n-2} - (n-2) D_{n-3}]$$

$$= (-1)^3 [D_{n-3} - (n-3) D_{n-4}]$$

$$(-1)^{n-2} [D_2 - 2D_1]$$

$$\text{But } D_2 = 1 \quad D_1 = 0 \quad \text{so}$$

$$D_n - n D_{n-1} = (-1)^{n-2} \quad \text{or}$$

$$D_n = n D_{n-1} + (-1)^{n-2}$$

$$\Rightarrow D_n = n D_{n-2} + (-1)^n$$

$$\text{Ex } D_7 = 7 D_6 + (-1)^7 = 7 \cdot 265 - 1$$

$$= 1854$$

12

Ex. At a party there are  $n$  men and  $n$  women. How many ways can the women choose a male dance partner?

Ans  $n!$

How many ways can a second partner be chosen if there are no repetitions?

Ans  $D_n$

Ex Determine the number of permutations of  $\{1, \dots, 9\}$  in which at least one odd integer is in its natural position

- $D_1$  1 in natural position
- $D_3$  3 in natural position
- $D_5$  5 in natural position
- $D_7$  7 in natural position
- $D_9$  9 in natural position

$\bar{D}_1, \dots, \bar{D}_9$  no odd in natural position

$S$  = all permutations,

| At least one odd | =  $|S| - \text{no odds in natural position}$

$$|D_1| = 8! = |D_3| = |D_5| = |D_7| = |D_9|$$

$$\text{Total } 8! 5$$

$$|D_1 \cap D_3| = 6! \left(\begin{array}{c} 5 \\ 2 \end{array}\right)$$

$$|D_1 \cap D_3 \cap D_5| = 6! \left(\begin{array}{c} 3 \\ 3 \end{array}\right)$$

$$|D_1 \cap D_3 \cap D_5 \cap D_7| = 5! \left(\begin{array}{c} 5 \\ 5 \end{array}\right)$$

$$|D_1 \cap D_3 \cap D_5 \cap D_7 \cap D_9| = 4! \left(\begin{array}{c} 5 \\ 5 \end{array}\right)$$

$$\begin{aligned} (\text{At least one odd}) &= |S| - |\bar{D}_1 \cap \bar{D}_3 \cap \bar{D}_5 \cap \bar{D}_7 \cap \bar{D}_9| \\ &= |S| - |S| + 5 \cdot 8! - \left(\begin{array}{c} 5 \\ 2 \end{array}\right) 7! + \left(\begin{array}{c} 5 \\ 3 \end{array}\right) 6! \\ &\quad - \left(\begin{array}{c} 5 \\ 4 \end{array}\right) 5! + \left(\begin{array}{c} 5 \\ 5 \end{array}\right) 4!, \end{aligned}$$

14

Homework

Ch. ~~10~~ 6

11, 12, 14, 15, 16, 21