

MA 416

Lesson 19

# FORBIDDEN POSITIONS

Let  $S = \{1, \dots, n\}$  There are  $n!$  permutations of  $S$  where  $L_1 \dots L_n$  means  $1 \rightarrow L_1, \dots, n \rightarrow L_n$ .

Given subsets  $X_1, \dots, X_n$  of  $S$

we consider all permutations

where  $L_1 \notin X_1, \dots, L_n \notin X_n$

Hence  $L_1 \in \bar{X}_1, \dots, L_n \in \bar{X}_n$  where

$\bar{X}$  is the complement of  $X$

Ex  $S = \{1, 2, 3, 4\}$

$X_1 = \{1, 3\}$     $X_2 = \{2, 3\}$     $X_3 = \{4\}$     $X_4 = \emptyset$

$\bar{X}_1 = \{2, 4\}$     $\bar{X}_2 = \{1, 4\}$     $\bar{X}_3 = \{1, 2, 3\}$     $\bar{X}_4 = \{1, 2, 3, 4\}$

Let  $P(X_1, \dots, X_n)$  be all permutations of the type in the first paragraph

Ex In the example we have

$P(X_1, X_2, X_3, X_4)$ . Since  $|S|$  is small

we can find  $P(X_1, X_4)$  by listing

all permutations of  $S$ . We list

in order where  $1, 2, 3, 4$  so

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1234	2134	3124	4123
1324	2143	3142	4132
1243	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

Since  $1 \in X_1$ , the first column can't be used  
 Since  $3 \in X_1$ , the third column can't be used  
 Since  $X_2 = \{2, 3\}$ , the middle 2 elements in  
 column 2 are eliminated as are the last  
 4 elements in column 4. Since  $4 \in X_3$   
 the second element in column 2 is removed  
 We are left with

2134, 2413, 2431, 4123, 4132

$$\text{So } P(X_1, X_2, X_3, X_4) = |P(X_1, X_2, X_3, X_4)| = 5$$

$\in X$  let  $S = \{1, \dots, n\}$  and  $X_1 = \{1\}, X_2 = \{2\},$   
 $\dots$   $X_n = \{n\}$ . The elements which can  
 not be used  $(1,1) \dots (n,n)$  are exactly  
 those that give a derangement on  $S$

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When  $n$  is larger, The possibilities grow. We consider The problem

geometrically, using a  $n \times n$  chess board and non-attacking rooks

Each permutation of  $1, \dots, n$  gives a non-attacking rook arrangement

and conversely. Label The rows  $1, \dots, n$  and assign The columns as a permutation of  $1, \dots, n$ . This is a non-attacking rook positioning

Set up The board by placing  $x$  in the  $(i, j)$  position if  $j \in X_i$

In our example  $1, 3 \in X_1$ ,  $2, 3 \in X_2$ ,  $4 \in X_3$

giving

x		x	
	x	x	
			x

We are looking for all non-attacking rook arrangements such that the patterns are used in the inclusion-exclusion formula

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The placement of the  $n$  non-attacking rooks satisfies  $P_1$  if the rook in row  $j$  is in a column belonging to  $x_j$  ( $x$  has an  $x_j$ )  $A_j$  is all rook placements satisfying  $P_1$ .  $P(x_1, \dots, x_n)$  consists of all placements of non-attacking rooks that satisfy none of  $P_1, \dots, P_n$ . So,

$$P(x_1, \dots, x_n) = |\bar{A}_1 \cap \dots \cap \bar{A}_n| = n! - (|A_1| + \dots + |A_n|) + (\sum |A_i \cap A_j|) - (\sum |A_i \cap A_j \cap A_k|) \dots + (-1)^n |A_1 \cap \dots \cap A_n|$$

$|A_1|$  is the number of ways of placing  $n$  non-attacking rooks on the board where rook in row 1 is in one of the columns with an  $x$ . This can be chosen in  $|x_1|$  ways and then the rest of the rows can be filled out in  $(n-1)!$  ways

$$|A_1| = |x_1| (n-1)!$$

Then  $\sum |A_i| = |x_1| (n-1)! + \dots + |x_n| (n-1)!$   
 $= (|x_1| + \dots + |x_n|) (n-1)!$

Let  $r_1 = |x_1| + \dots + |x_n|$  be the number of  $x$ 's on the board

$|A_1 \cap A_2|$ : The number of ways of picking X's in rows 1 and 2 so they start a permutation (can't be in the same column). Then fill out the board in  $(n-2)!$  ways. The same holds for  $A_1 \cap A_j$ . If  $r_2$  is the number of ways of picking X's so they are not in the same column

$$\sum |A_1 \cap A_j| = r_2 (n-2)!$$

For  $k, A_k$ 's

$$\sum |A_{i_1} \cap \dots \cap A_{i_k}| = r_k (n-k)!$$

and  $r_k$  is the number of ways of picking  <sup>$\setminus$</sup>  X's on the board so they are not in the same column for every pair

Hence  $|\bar{A}_1 \cap \dots \cap \bar{A}_n| =$

$$n! - n(n-1)! + r_2(n-2)! - \dots \quad (-1)^n r_n$$

= permutations each of whose elements avoid ~~the~~ forbidden positions (i.e. the answer we want)

Notice that  $r_1$  is just the number of X's on the board  
 For the others, we need to find all combinations where no 2 X's are in the same column.

It is much easier if the X's are in collections where the columns in the first collection and the columns in the second collection do not overlap.

Ex Gains back to example

$$S = \{1, 2, 3, 4\}$$

$$X_1 = \{1, 2\} \quad X_2 = \{2, 3\} \quad X_3 = \{4\} \quad X_4 = \emptyset$$

$$\begin{pmatrix} X & & & \\ & X & & \\ & & X & \\ & & & X \end{pmatrix}$$

$$|S| = 4! = 24$$

$$r_2 = 5$$

For  $r_2$  blocks The matrix as

$$\begin{pmatrix} X & & X \\ \hline & X & X \\ & & & X \end{pmatrix}$$

or  $\frac{1}{2}$  from block 1 and none from block 2

We can either pick any of 4 from block 1 and 1 from block 2

?  
 $r_2$ : 2 from block 1 could be, using their coordinates:

(1, 1) (2, 3)      Total 3

(1, 1) (2, 3)

(1, 3) (2, 2)

1 from block 1, 1 from block 2

(1, 1) (3, 4)

(1, 3) (3, 4)

(2, 2) (3, 4)

(2, 3) (3, 4)

Total 4

$$r_2 = 3 + 4 = 7$$

$r_3$  There must be 2 from block 1  
 1 from block 2

(1, 1) (3, 4) (1, 1) (2, 2) (3, 4)

(1, 3) (3, 4)

(1, 1) (2, 3) (3, 4)

(1, 3) (2, 4) (3, 4)

$$r_3 = 3$$

$$r_4 = 0$$

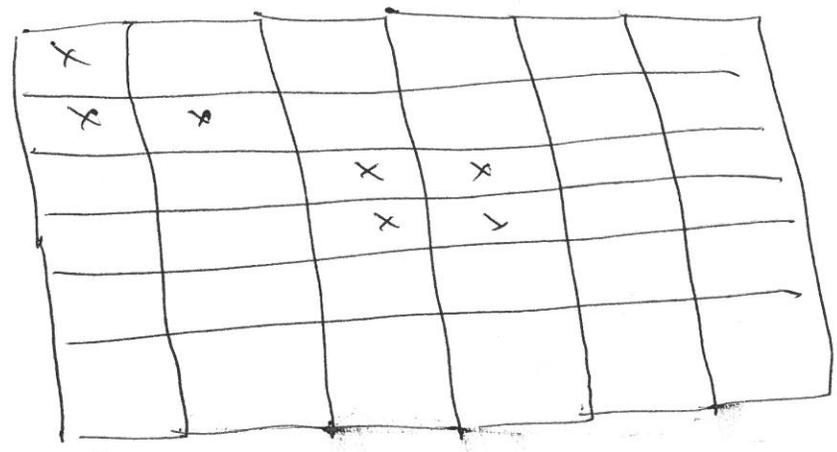
$$P(x_1, x_2, x_3, x_4) = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| =$$

$$4! - 5 \cdot 3! + (3+4)2! - 3!$$

$$= 24 - 30 + 14 - 3 = 5$$

which matches with the calculation on page 2, done by listing the permutations

Example Find the number of ways to place 6 nonattacking rooks on the following 6x6 board



This is the same as the number of

permutations on  $S = \{1, 2, 3, 4, 5, 6\}$

with  $X_1 = \{1\}$   $X_2 = \{1, 2\}$   $X_3 = \{3, 4\}$   $X_4 = \{3, 4\}$

being restrictions on the permutations

$r_1 = 7$

$r_2: (1,1) (2,2)$

$(3,3) (4,4), (3,4) (4,3)$

$(1,1) (3,3) (1,1) (3,4), (1,1) (4,3), (1,1) (4,4)$

$(2,2) (3,3) (2,2) (3,4) (4,4) (4,3), (2,4) (4,4)$

$(2,2) (3,3) (2,2) (3,4) (2,2) (4,3) (2,2) (4,4)$

$r_2 = 15$

$r_3$   ~~$(1,1) (3,3) (1,1) (4,3) (1,1)$~~

$(1,1) (3,3) (4,4), (1,1) (3,4) (4,3)$

$(2,1) (3,3) (4,4), (2,1) (3,4) (4,3)$

$(2,2) (3,3) (4,4), (2,2) (3,4) (4,3)$

$(1,1) (2,1) (3,3), (1,1) (2,2) (3,4), (1,1) (2,4) (4,3), (1,1) (2,2) (4,4)$

$r_3 = 10$

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 $r_4$ 

(11)(22)(33)(44) (11)(22)(34)(43)

$$r_4 = 2$$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = 6! - r_1 5! + r_2 4!$$

$$- r_3 3! + r_4 2! - r_5 1! + r_6 0!$$

$$= 6! - 7 \cdot 5! + 15 \cdot 4! - 10 \cdot 3! + 2 \cdot 2!$$

$$+ 0 \cdot 1! + 0 \cdot 0!$$

Problems

Chapter 6: 24, 25, 26