

M A 416
Lesson 20

Problem 12. Find the number
of permutations of $\{1, \dots, 8\}$
in which exactly 4 integers are
in their natural position

First, we pick the 4 in $\binom{8}{4}$ ways

Then the other 4 can not be
in their natural position, D_4 ways

$$\text{Answer} = \binom{8}{4} D_4$$

Problem 16. Show

$$n! = \binom{n}{0} D_n + \binom{n}{1} D_{n-1} + \dots + \binom{n}{n} D_0$$

If exactly j elements are in
their natural position, this can be done
in $\binom{n}{j} D_{n-j}$ ways. The set of all
permutations is partitioned into
sets with exactly j elements fixed
 $j=0, \dots, n$. The number in each of
these sets is the number in the
terms in the right hand side
of the equation *. This gives
the result

Problem 28. A carousel has 8 seats each representing a different animal. Eight boys are seated in the 8 seats and are facing inward (There are 4 pairs across from each other) In how many ways can the boys change seats so that each faces a different boy?

We use inclusion-exclusion
Let A_1 be the arrangements where pair 1 face each other

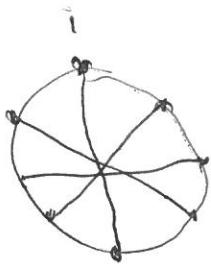
A_2, A_3 and A_4 are similar

The answer to the question is

$$\begin{aligned} |A_1 \cap A_2 \cap A_3 \cap A_4| &= 15! - (|A_1| + |A_2| + |A_3| + |A_4|) \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &\quad + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) - |A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

If the seats are on the same this answer is divided by 4

3



For A₁, There are (4) pairs of seats for the 1 pair to sit. Then 2 choices for seating them Total = $4 \cdot 2 = 8$

S. Then the remaining boys can be arranged in 6! ways. Total = $4 \cdot 2 \cdot 6! = 1A_1, 1$

$$\text{For } A_1, A_2: 1A_1 + 1A_2 + 1A_3 + 1A_4 = 4 \cdot [4 \cdot 2 \cdot 6!]$$

of seats. Then 4 ways of arranging the boys in the chosen seats. Then 4! ways to seat the other boys. There are $\binom{4}{2}$ choices of pairs A₁, A₂. The total is $\binom{4}{2} \binom{4}{2} 4 \cdot 4!$ There are $\binom{3}{2}$ ways of picking pairs of seats for the boys. Total $\binom{4}{2} \binom{4}{2} \binom{3}{2} 4 \cdot 4!$

For A₁A₂A₃: $\binom{4}{3}$ choices for 3 sets of boys
 $\binom{4}{3}$ choices for 3 sets of seats
 $3!$ ways of assigning pairs of boys
 to pairs of seats 2 · 2 · 2 of pairing boys
 into seats and 2! ways for last 2 boys

4
For $A_1 \cap A_2 \cap A_3 \cap A_4$

(4) choices of pairs of boys

(4) choices of pairs of seats

$4!$ ways of putting pairs of boys into pairs of seats

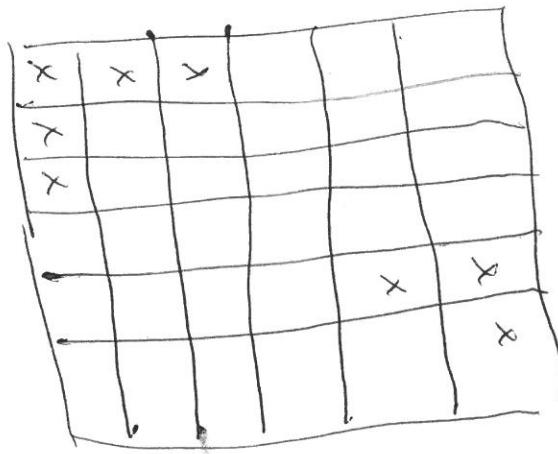
$2 \cdot 2 \cdot 2 \cdot 2$ ways to pick the two seats for each pair of boys

$$\text{Total.} = 4! \cdot 2^4$$

$$\begin{aligned} \text{Ans} &= 8! - (\text{?})(\text{?}) 2 \cdot 6! + (\text{?})(\text{?}) \cancel{\frac{2}{2}} 4 \cdot 4! \\ &- (\text{?})(\text{?}) 3! \cdot 2 \cdot 2 \cdot 2! \\ &+ (\text{?})(\text{?}) 4! \cdot 2^4 \end{aligned}$$

Problem 38

Fill in forbidden positions



$$r_1 = \text{total } x's = 8$$

$$r_2 = (2 \text{ from block 1}) (0 \text{ from block 2})$$

$$+ (1 \text{ from block 1}) (1 \text{ from block 2})$$

$$+ (0 \text{ from block 1}) (2 \text{ from block 2})$$

$$= 4 + 5 \cdot 3 + 1 = 20$$

$$r_3 = (3 \text{ from 1}) (0 \text{ from 2}) + (2 \text{ from 1}) (1 \text{ from 2})$$

$$+ (1 \text{ from 1}) (2 \text{ from 2}) + (0 \text{ from 1}) (3 \text{ from 2})$$

$$= 0 + 4 \cdot 3 + 5 \cdot 1 + 0 = 17$$

$$r_4 = (4 \text{ from 1}) (0 \text{ from 2}) + (3 \text{ from 1}) (1 \text{ from 2})$$

$$+ (2 \text{ from 1}) (2 \text{ from 2}) + (1 \text{ from 1}) (3 \text{ from 2})$$

$$+ (1 \text{ from 1}) (4 \text{ from 2})$$

$$+ (0 \text{ from 1}) (5 \text{ from 2}) = 0$$

$$= 0 + 0 + 4 \cdot 1 + 0 + 0 = 4$$

$$r_5 = r_6 = 0$$

$$(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5 \cap \bar{A}_6) = 6! - 8 \cdot 5! + 20 \cdot 4!$$

$$- 17 \cdot 3! + 4 \cdot 2!$$

6 More Forbidden positions
The numbers 1,..,8 are permuted such that no number follows the same number as when we start so 6 does not follow 5

In total, the patterns

12 23 34 45 56 67 78

do not appear

Let Q_8 be this number.

More generally Q_n stands for the permutations of 1,..,n where ~~not~~ s+1 does not follow s, $s=1, \dots, n-1$

$$Q_2: 2, \quad Q_3: 132 \quad Q_2 = 1 \\ \begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 3 \end{matrix} \quad Q_3 = 2 \\ Q = 1$$

Show $Q_4 = 11$ by listing all permutations of 1,2,3,4 and finding the ones with the condition

$$\text{Theorem. } Q_n = n! - \binom{n-1}{1}(n-1)! \\ + \binom{n-1}{2}(n-2)! \dots + (-1)^{\binom{n-1}{n-1}} 1!$$

From the form of the theorem it should be clear we will use inclusion-exclusion

$S = \text{all } n!$ permutations

P_j , property $j(j+1)$ does not appear

A permutation counts in Q_n iff
 P_1, \dots, P_{n-1} does not hold in it.

$A_j = \text{all permutations satisfying } P_j$

Then $Q_n = |A_1 \cap A_2 \cap \dots \cap A_{n-1}|$

A permutation in A_i can be viewed

as a permutation of $1, 2, 3, \dots, n$

i, j are taken together so there are

$n-1$ elements in the permutation's set

$\therefore |A_i| = (n-1)!$ Likewise $|A_j| = (n-1)!$

For $A_1 \cap A_2$, i, j, k are taken

together, $|A_1 \cap A_2| = (n-2)!$

For $A_1 \cap A_2 \cap A_3$ i, j, k occur together

and the set can be viewed as

$\{1, 2, 3, 4, \dots, n\}$ $n-3$ elements Again there

are $(n-3)!$ permutations For all

cases $|A_1 \cap A_2 \cap A_3| = (n-3)!$

Likewise the intersection of k

A_i has $(n-k)!$ permutations

Pairing k pairs from $n-1$ elements

can be done in $\binom{n-1}{k}$ ways.

$$\text{Hence } \Phi_n = n! - \binom{n-1}{1} (n-1)! + \binom{n-1}{2} (n-2)!$$

$$+ \dots + (-1)^{n-1} \binom{n-1}{n-1} 1!$$

$$\Phi_5 = 5! - (4)4! + (3)3! - (2)2! + (1)1!$$

$$= 53$$

Problems Chapter 6

24, 25, 27, 28, 29