

MA 416

Lesson 21

Fibonacci Numbers

FIBONACCI SEQUENCE

In 1202, Liber Abaci published the following problem.

A newly born pair of rabbits of opposite sexes is born. Beginning every second month, each month they have a new pair of rabbits, again of opposite sexes. The new rabbits follow the same pattern, as do their offspring. How many rabbits are there after a year?

Let f_n be the number of pairs of rabbits at the beginning of month n . So $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$.

At the beginning of month n there are two groups of rabbits, those present at the beginning of month $n-1$ and those present born during month $n-1$. So at the beginning of month n , $f_n = f_{n-1} + f_{n-2}$. This is an example of a recurrence relation.

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We use it to compute some f_n

$$f_5 = f_4 + f_3 = 3 + 2 = 5$$

$$f_6 = f_5 + f_4 = 5 + 3 = 8$$

$$f_7 = f_6 + f_5 = 8 + 5 = 13$$

The next few values are $f_8 = 21$, $f_9 = 34$

$$f_{10} = 55 \quad f_{11} = 89 \quad f_{12} = 144 \quad f_{13} = 233, \text{ So}$$

after 1 year there are $f_{13} = 233$

pairs of rabbits

Let $f_0 = 0$, $f_1 = 1$. Then f_0, f_1, \dots, f_n is

a sequence satisfying $f_n = f_{n-1} + f_{n-2}$

With initial conditions $f_0 = 0$, $f_1 = 1$

The sequence is called a Fibonacci sequence.

Lemma. The partial sum

$$S_n = f_0 + f_1 + \dots + f_n \quad \text{has} \quad S_n = f_{n+2} - 1$$

Proof Induct on n

$$\text{For } n=0, \quad 0 = f_0 = f_2 - 1 = 1 - 1 = 0 \quad (f_0 < S_0)$$

Assume the result holds for n . Then

$$S_{n+1} = f_0 + f_1 + \dots + f_{n+1} = (f_0 + \dots + f_n) + f_{n+1} =$$

$$(f_{n+2} - 1) + f_{n+1} = f_{n+3} - 1$$

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Lemma $2 \mid f_n$ iff $3 \mid n$

Proof Write out some f_n

$$f_0 = 0 \quad f_1 = 1 \quad f_2 = 1 \quad f_3 = 2 \equiv 0 \pmod{2}$$

$$f_4 = f_2 + f_3 \pmod{2} = 3 \pmod{2} \equiv 1$$

$$f_5 = f_3 + f_4 \pmod{2} = 2 + 1 \equiv 1$$

$$f_6 = f_4 + f_5 \pmod{2} = 1 + 1 \equiv 0$$

We see the pattern and assume that

$$f_{3k-2} \equiv 1 \pmod{2}$$

$$f_{3k-1} \equiv 1 \pmod{2}$$

$$f_{3k} \equiv 0 \pmod{2}$$

Then show

$$f_{3(k+1)-2} = f_{3k-1} + f_{3k} \equiv 1 + 0 \pmod{2} \equiv 1$$

$$f_{3(k+1)-1} = f_{3k} + f_{3k+1} \equiv 0 + 1 \pmod{2} \equiv 1$$

$$f_{3(k+1)} = f_{3k+2} + f_{3k+1} \equiv 1 + 1 \pmod{2} \equiv 0$$

So the result holds

We used $\pmod{2}$ since we were considering

when $2 \mid f_n$ i.e. $f_n \equiv 0 \pmod{2}$

We have f_n as a recurrence formula. Can we get f_n in closed form? Yes

$$\text{Consider } f_n - f_{n-1} - f_{n-2} = 0$$

Let $f_n = q^n$, $q \neq 0$. Then

$$q^n - q^{n-1} - q^{n-2} = 0$$

$$q^{n-2} [q^2 - q - 1] = 0$$

$$q^2 - q - 1 = 0$$

$$q = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n, \left(\frac{1-\sqrt{5}}{2}\right)^n$$

All solutions are linear combinations of these two solutions:

$$f_n = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Use initial values $f_0 = 0$ $f_1 = 1$ So,

$$0 = C_1 + C_2$$

$$1 = C_1 \left(\frac{1+\sqrt{5}}{2}\right) + C_2 \left(\frac{1-\sqrt{5}}{2}\right) \rightarrow$$

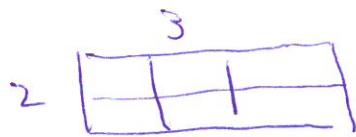
$$C_1 = \frac{1}{\sqrt{5}}, C_2 = -\frac{1}{\sqrt{5}} \text{ Hence}$$

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

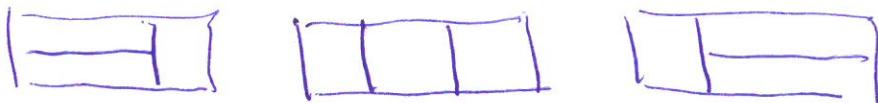
is the closed form solution

More Results having

Fibonacci Numbers

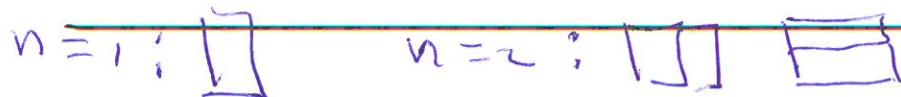


A 2 by 3 board. How many ways can it be covered by dominoes?



3 ways

Let h_n be the number of ways for an $2 \times n$ board

$$h_1 = 1 \quad h_2 = 2 \quad h_3 = 3$$


Define $h_0 = 1$

Let $n \geq 2$

Let A be the covers where the first domino is vertical . The rest

can be covered in h_{n-1} ways

Let B be the covers where the first domino is horizontal

Then is the first 2×2 . The

rest can be covered in h_{n-2} ways

$$\text{So } h_n = h_{n-1} + h_{n-2} \quad n \geq 1, h_0 = 1, h_1 = 2$$

Thus we get the Fibonacci numbers, $h_n = f_n$. This holds not only because $h_n = h_{n-1} + h_{n-2}$ but also the initial conditions

Ex Determine the number of covers of a $1 \times n$ board with monomials and dominoes. Each board has a corresponding cover of a $2 \times n$ board with dominoes. So again we get the Fibonacci numbers. If b_n is the number then

$$b_n = h_n = f_n,$$

Recall the Pascal Triangle

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Start in left diagonal. Reportedly, let the upper element in the binomial decrease by one and the lower element increase by 1. The sum of these is a Fibonacci number

$$f_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \dots + \binom{n-t}{t}$$

$$t = \left\lfloor \frac{n+1}{2} \right\rfloor = \text{last non zero term}$$

$$f_5 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5$$

$$f_6 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 8$$

Actually, you write out binomial coefficients until the top number is less than the bottom number

Proof

$$\text{Let } g_n = \binom{n+1}{0} + \binom{n-2}{1} + \dots + \binom{n-t}{t-n}$$

as in the theorem.

Then

$$g_n = \binom{n-1}{0} + \binom{n-2}{1} + \dots + \binom{0}{n-1} \text{ since}$$

all the new terms are 0

$$\text{So } g_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$$

$$\begin{aligned} g_{n-1} + g_{n-2} &= \sum_{k=0}^{n-2} \binom{n-2-k}{k} + \sum_{j=0}^{n-3} \binom{n-3-j}{j} \\ &= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-2-k}{k} + \sum_{k=1}^{n-2} \binom{n-2-k}{k-1} \\ &= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-2-k}{k} + \binom{n-2-k}{k-1} \quad (\text{Let } j = k-1) \\ &= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-1-k}{k} \quad (\text{Pascal's Identity}) \\ &= \binom{n-1}{0} + \sum_{k=1}^{n-1} \binom{n-1-k}{k} + \binom{0}{n-1} = \sum_{k=0}^{n-1} \binom{n-1-k}{k} = g_n \end{aligned}$$

$$\text{So } g_n = g_{n-1} + g_{n-2} \stackrel{\text{def}}{=} g_2 = \binom{1}{0} + \binom{0}{1} = 1$$

$g_0 = \binom{0}{0} = 0$ $g_1 = \binom{0}{0} = 1$ initial conditions for Fibonacci

$$\text{The initial conditions for Fibonacci} \quad f_n = g_n$$

Hence

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* Ex Write $f_0 + f_2 + \dots + f_{2n}$ in closed form

$$f_0 + f_2 = 0 + 1 = 1$$

$$f_0 + f_2 + f_4 = 0 + 1 + 3 = 4$$

$$f_0 + f_2 + f_4 + f_6 = 0 + 1 + 3 + 5 + 8 = 12$$

$$f_0 + \dots + f_6 + f_8 = 0 + 1 + 3 + 5 + 8 + 13 = 33$$

$$\text{Now } f_1 = 1 \quad f_2 = 1 \quad f_3 = 2 \quad f_4 = 3 \quad f_5 = 5$$

$$f_6 = 8 \quad f_7 = 13 \quad f_8 = 21 \quad f_9 = 34$$

We see in the examples

* $f_0 + \dots + f_{2n} = f_{2n+1} - 1$

Use induction

$$f_0 + f_2 = 0 + 1 = 1 = f_3 - 1 = 2 - 1$$

Assume * Then

$$f_0 + \dots + f_{2n} + f_{2n+2} = (f_{2n+1} - 1) + f_{2n+2}$$

$$= f_{2n+3} - 1$$

So the result holds

Problems Ch 7

1, 3, 4, 5, 8, 9