

mA 4/6

Lesson 22

1
We will introduce generating functions.

First, recall that if

$$e_1 + \dots + e_r = n \quad e_i \geq 0$$

The number of non negative integer solutions is

$$\binom{n+r-1}{r-1}$$

Recall from calculus that

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

From that,

$$\left(\frac{1}{1-x}\right)^r = (1+x+\dots)(1+x+\dots)\dots(1+x+\dots) \\ = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

c_n is the number of ways of combining the products to get c_n , e_1 from the first series, e_2 from the second series, ... e_r from the last series with $x^{e_1} \dots x^{e_r} = x^n$

The number of ways of doing this is

$$e_1 + \dots + e_r = n \quad e_i \geq 0$$

$$\text{So } \frac{1}{(1-x)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} x^n$$

$g(x) = \frac{1}{(1-x)^r}$ is called the generating

2 function for the problem, more generally, given a combinatorial counting problem which has a value a_n for each non-negative integer n , the generating function $g(x)$ for the problem is a Taylor series $\sum_{n=0}^{\infty} a_n x^n$ such that a_n is the number of ways solutions to the problem.

Ex $\sum_{n=0}^{\infty} f_n x^n$ is the generating function for the Fibonacci sequence

Ex Let r be an integer. Let h_n be the number of non-negative solutions to

$$e_1 + \dots + e_k = n$$

then $a_n = \binom{n+r-1}{r-1}$ and the

generation function is

$$\sum \binom{n+r-1}{r-1} x^n = \frac{1}{(1-x)^r}$$

as we have just seen

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Ex. Find the generating function for the number of n combinations of apples, bananas, oranges and pears if the number of apples is even the number of bananas is odd the number of oranges is between 0 and 4

there is at least one pear

We are looking for the number of solutions to

$$e_1 + e_2 + e_3 + e_4 = n$$

where e_1 is even

e_2 is odd

e_3 is between 0 and 4

$$e_4 \geq 1$$

The corresponding power series

$$1 + x^2 + x^4 + \dots + x^{2n} \text{ represents}$$

the even condition. Similar

remarks for the rest give

$$(1 + x^2 + x^4 + x^6 + \dots) (x + x^3 + x^5 + x^{2n+1}) (1 + x + x^3 + x^5)$$

$$(x + x^3 + x^5 + x^7 + \dots) = \frac{1}{1-x^2} \frac{x}{1-x^2} \frac{1-x^5}{1-x} \frac{x}{1-x}$$

$$= \frac{x(1-x^5)}{(1-x^2)^2(1-x)} = g(x)$$

If this $g(x)$ is expanded in a Taylor series, the coefficient of x^n is the number of ways n pieces of fruit can be chosen.

Ex. Another fruit example.

Number of apples is even

Number of bananas is a multiple of 5

Number of oranges is at most 4

Number of pears is 0 or 1

Solution

$$\begin{aligned} g(x) &= (1+x^2+x^4+\dots)(1+x^5+x^{10}+\dots) \\ &\quad (1+x^2+x^3+x^5+x^8)(1+x) \\ &= \frac{1}{1-x^2} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^5}{1-x} (1+x) = \end{aligned}$$

$$\begin{aligned} \frac{1+x}{(1-x^2)(1-x)} &= \frac{1}{(1-x)^2} = \sum \binom{n+1}{n} x^n \\ &= \sum (n+1)x^n \end{aligned}$$

$$\text{To see } \frac{1}{(1-x)^2} = \sum \binom{n+1}{n} x^n \text{ see}$$

page 1 of these notes

Ex Find the generating function
for the sequence $0, 1, 2, 3, \dots$

$$g(x) = \sum_{n=0}^{\infty} nx^n \quad \text{using } a_n = n$$

$$\text{From } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} \quad \text{by}$$

differentiation. Then

$$\left(\frac{x}{1-x}\right)^2 = \sum_{n=0}^{\infty} nx^n = g(x)$$

$$g(x) = \frac{x}{(1-x)^2}$$

Note that if each series
has only a finite number of
terms, then $g(x)$ will be a
polynomial

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Find the generating function for the number of non-negative solutions to

$$3e_1 + 4e_2 + 2e_3 + 5e_4 = n$$

Start with a change of variables

$$f_1 = 3e_1, \quad f_2 = 4e_2, \quad f_3 = 2e_3, \quad f_4 = 5e_4$$

so we want the non-negative solutions to $f_1 + f_2 + f_3 + f_4 = n$

where f_i is a multiple of 3

f_2 is a multiple of 4

f_3 is a multiple of 2

f_4 is a multiple of 5

The generating function comes from

$$(1+x^3+x^6+\dots)(1+x^4+x^8+\dots)(1+x^2+x^4+\dots)(1+x^5+x^{10}+\dots)$$

$$= \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^5} = g(x)$$

Ex There are unlimited numbers of pennies, nickles, dimes, quarters and half dollars. Find the generating function $g(x)$ for the number h_n of ways of making up n cents using these coins.

$$e_1 + 5e_2 + 10e_3 + 25e_4 + 50e_5 = n$$

$$\text{let } f_1 = e_1, \quad f_2 = 5e_2, \quad f_3 = 10e_3$$

$$f_4 = 25e_4, \quad f_5 = 50e_5$$

Then

$$f_1 + f_2 + f_3 + f_4 + f_5 = n, \quad f_i \geq 0$$

and each is a certain multiple
so

$$g(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}} \cdot \frac{1}{1-x^{50}}$$

Homework Chapter 7

14, 15, 16, 17, 18, 19, 20

Problem 1 d on page 257

$$f_0^2 + f_1^2 + \dots + f_n^2 = ?$$

$$f_0^2 = 0$$

$$f_0^2 + f_1^2 = 1$$

$$f_1^2 + f_2^2 = 2$$

$$f_1^2 + f_2^2 + f_3^2 = 1 + 1 + 4 = 6$$

$$f_1^2 + f_2^2 + f_3^2 + f_4^2 = 1 + 1 + 4 + 9 = 15 = f_4 \cdot f_5$$

$$f_1^2 + f_2^2 + \dots + f_5^2 = 1 + 1 + 4 + 9 + 25 = 40 = f_5 \cdot f_6$$

$$\text{Prove } f_1^2 + \dots + f_n^2 = f_n f_{n+1}$$

Assume this and consider

$$f_1^2 + \dots + f_n^2 + f_{n+1}^2 =$$

$$f_n f_{n+1} + f_{n+1}^2 = (f_n + f_{n+1}) f_{n+1}$$

$$= f_{n+1} f_{n+2} \text{ shows the}$$

induction step

$$\text{Also } f_1^2 + f_2^2 = 2$$

$$+_2 f_3 = 1 \cdot 2 = 2$$

Shows the first step

Number 8 Ch. 7

E_x h_n = number of ways we can color a $1 \times n$ board red or blue with no 2 squares adjacent being red

$$n=1 \quad \begin{array}{c} \square \\ \text{R} \\ \square \\ \text{B} \end{array} \quad 2 = h_1$$

$$n=2 \quad \begin{array}{c} \square \square \\ \text{R} \text{ B} \\ \square \square \\ \text{B} \text{ B} \\ \square \square \\ \text{B} \text{ R} \end{array} \quad h_2 = 3$$

$$n=3 \quad \begin{array}{c} \square \square \square \\ \text{R} \text{ B} \text{ B} \\ \square \square \square \\ \text{R} \text{ B} \text{ R} \\ \square \square \square \\ \text{B} \text{ R} \text{ B} \end{array} \quad h_3 = 5$$

To construct the number for $n=4$

Each $n=3$ case can add a blue

The only $n=3$ cases which can add a red are those which end with blue = the total number in $n=2$ case

So if ~~total~~ number h

$$\text{So } h_4 = h_3 + h_2$$

The same pattern holds for n

The $n-1$ cases which can add blue are all of them, h_{n-1} . The ones which can add red are the ones which end with blue = Total in $n-2$ case = h_{n-2} So

$$h_n = h_{n-1} + h_{n-2} \quad h_1 = 2 \quad h_2 = 3 \quad h_3 = 5$$

11 We have obtained a shifted Fibonacci sequence

$$h_n = f_{n+2}$$

Since $h_n = h_{n-1} + h_{n-2}$ $h_1 = 2$, $h_2 = 3$

and $f_n = f_{n-1} + f_{n-2}$ $f_3 = 2$, $f_4 = 3$

So $h_3 = f_5 = 5$

$$h_4 = f_6 = 8$$

$$h_5 = f_7 = 13$$

$$h_6 = f_8 = 21$$