

mA 416

Lesson 23

Review of generating functions.

For each n , let h_n be the number of ways to complete a certain task. Let

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

$g(x)$ is called the generating function for the task. This power series is viewed differently than what we saw in calculus. The x^n are placeholders for the h_n and we do not plug values for x into it. We do not worry about convergence.

Ex For a fixed r , let h_n be the number of solutions to

$$e_1 + \dots + e_r = n \quad e_i \geq 0$$

Then $h_n = \binom{n+r-1}{r-1}$ and

$$g(x) = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} x^n$$

Note that this problem is a multiset combination problem. It is the kind of problem that we use these power series for. The geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

is an example where

$h_n = 1$. We also have written the series as a simple function

$$\text{Ex } \frac{1}{(1-x)^k} = (1+x+\dots+x^{n_1}+\dots)(1+x+\dots+x^{n_2}+\dots)(1+x+\dots+x^{n_3}+\dots)\dots(1+\dots+x^{n_r}+\dots)$$

The coefficient of x^n in the product is the number of ways of computing $x^{n_1}\dots x^{n_r}$ where

$$n_1 + \dots + n_r = n, \quad n_i \geq 0$$

This is $\binom{n+r-1}{r-1} = h_n$, so the series

$$\sum_{n=0}^{\infty} \binom{n+r-1}{r-1} x^n = \frac{1}{(1-x)^r}$$

Last time we saw variations
of the problem. Perhaps all the
first terms needed to be even,
all the second terms needed
to be odd, the third terms were
all less than 6. Then

$$(1+x^2+x^4+\dots)(x+x^3+x^5+\dots)(1+x^2+x^4+\dots+x^6)$$
$$= \frac{1}{1-x^2} \cdot \frac{x}{1-x^2} \cdot \frac{x^6-1}{x-1} = g(x)$$

Then h_n is the term in the expansion
of $g(x)$

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We now look at another generating function, the exponential generating function.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the basic one.

These are used for multiset permutation problems.

Ex Suppose $S = \{x_1, x_2, x_3\}$. How many strings of n of these integers are there? Write

$$(1 + x + \frac{x}{1!} + \dots + \frac{x^{n_1}}{n_1!})(1 + x + \frac{x}{1!} + \dots + \frac{x^{n_2}}{n_2!})(1 + x + \frac{x}{1!} + \dots + \frac{x^{n_3}}{n_3!})$$

For $n = n_1 + n_2 + n_3$

$$h_n = \sum_{n_1+n_2+n_3=n} \frac{1}{n_1!} \frac{1}{n_2!} \frac{1}{n_3!}$$

$$g(x) = \sum_{n=0}^{\infty} \left(\sum_{n_1+n_2+n_3=n} \frac{1}{n_1!} \frac{1}{n_2!} \frac{1}{n_3!} \right) x^{n_1+n_2+n_3}$$

$$n=0 \quad n_1+n_2+n_3=n$$

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$$\frac{1}{n_1!} \frac{1}{n_2!} \frac{1}{n_3!} x^n = \frac{n!}{n_1! n_2! n_3!} \frac{x^n}{n!}$$

The coefficient is the number of permutations of n_1 's, n_2 's, n_3 's and we sum over all $n_1 + n_2 + n_3 = n$

$$h_n = \sum \binom{n}{n_1 n_2 n_3}$$

$$n_1 + n_2 + n_3 = n$$

Putting this together

$$\begin{aligned} e^x e^x e^x &= \left(1 + \frac{x^{n_1}}{n_1!} + \dots\right) \left(1 + \frac{x^{n_2}}{n_2!} + \dots\right) \left(1 + \frac{x^{n_3}}{n_3!} + \dots\right) \\ &= \sum h_n \frac{x^n}{n!} \\ &= e^{3x} = 1 + \frac{3x}{1!} + \frac{3^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \dots = \sum \frac{3^n x^n}{n!} \\ &= \sum h_n \frac{x^n}{n!} \rightarrow \\ h_n &= 3^n \end{aligned}$$

which is seen directly to be true

Ex Let h_n be the number of n digit numbers with digits 1, 2, 3 where the number of 1's is even, the number of 2's is at least 3 and the number of 3's is at least most 4

$$g(x) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) \left(\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right) \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)$$

$$= \frac{1}{2} [e^x + e^{-x}] [e^x - (1 + x + \frac{x^2}{2!})] \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)$$

$$h_3 = 1 \cdot 1 \cdot 1 = 1$$

$$h_4 = 1 \cdot 1 \cdot 1 + \cancel{1} = 1$$

Ex Find the number of ways to color a $1 \times n$ board with red and blue squares if
 if the number of blues is even and the number of reds
 is odd.

$$\begin{aligned}
 & \left(1 + \frac{x}{2!} + \frac{x^3}{4!} + \dots\right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \\
 &= \left(\frac{e^x + e^{-x}}{2}\right) \left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{4} (e^{2x} - e^{-2x}) \\
 &= \frac{1}{4} \left[1 + 2x + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots \right] \\
 &\quad - \frac{1}{4} \left[1 - 2x + \frac{(2x)^3}{3!} - \frac{(2x)^5}{5!} + \dots \right] = \\
 &\quad x + 2^2 \frac{x^3}{3!} + 2^4 \frac{x^5}{5!} + \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{So } h_n &= 2^{n-1} & n \text{ odd} \\
 &= 0 & n \text{ even}
 \end{aligned}$$

Homework ~~see~~ Ch. 7

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Problem 19. Let $h_n = \binom{n}{2}$. Find the generating function

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{Differentiate}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} \quad \text{Again}$$

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1) x^{n-2} \quad \rightarrow$$

$$\begin{aligned} \frac{1}{2} \frac{x}{(1-x)^3} &= \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n \\ &= \sum_{n=0}^{\infty} \binom{n}{2} x^n \end{aligned}$$

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Problem 9

h_n = number of ways to color
 a $1 \times n$ board, R B and W
 where no 2 R's are consecutive

$$n=1 \quad h_1 = 3$$

$$n=2 \quad h_2 = 8$$

To get the total colorings

for n, h_n : For any $n-1$ board
 we can add a W color (total h_{n-1})
 and add a B color (total h_{n-2})

We can add a R to any board
 that ends in B (total h_{n-2}) or
 ends in W (total h_{n-2}). In
 total $h_n = 2h_{n-2} + 2h_{n-1}$