

MA 416

Lesson 24

Linear Homogeneous Recurrence Relations

Def Let h_0, \dots, h_n be a sequence of numbers. They satisfy a linear recurrence relation of order k if there exist a_1, \dots, a_k with $a_k \neq 0$ and a quantity b_n such that

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \dots + a_k h_{n-k} + b_n$$

with $n \geq k$

Ex The Fibonacci numbers f_0, f_1, \dots

satisfy $f_n = f_{n-1} + f_{n-2}$

It is of order 2 with $a_1 = 1$

$$a_2 = 1 \quad b_n = 0$$

2

Ex Two recurrence relations
for the sequence of derangement
numbers, D_0, D_1, \dots, D_n are

$$D_n = (n-1) D_{n-1} + (n-1) D_{n-2} \quad (n \geq 2)$$

$$D_n = n D_{n-1} + (-1)^n \quad (n \geq 1)$$

The first has order 2, $a_1 = n-1 = a_2$
and $b_n = 0$

The second has order 1, $a_1 = n$ $b_n = (-1)^n$

Ex Geometric Sequence h_0, \dots, h_n

with $h_n = q^n$ satisfies

$$h_n = q h_{n-1}$$

This has order 1 $a_1 = q$ $b_n = 0$

3

The recurrence is called homogeneous if $b_n = 0$. It has constant coefficients if a_1, \dots, a_k are all constants.

Suppose

$$h_n = a_1 h_{n-1} + \dots + a_k h_{n-k} \quad \text{where}$$

the a_i are constants, write this as

$$h_n - a_1 h_{n-1} - \dots - a_k h_{n-k} = 0$$

We look for a sequence h_0, \dots, h_n, \dots that satisfies this relation.

The sequence will be uniquely determined if we are given initial values

$$h_0, \dots, h_{k-1}.$$

We will look at methods for solving for the h_i . The first method has a familiar analogue in differential equations

At times it is possible to solve a simple recurrence relation by looking at a few terms, making a guess at the formula and then showing it by induction.

Ex let $h_n = -h_{n-1} + 2$, $h_0 = 1$

Then

$$h_1 = -1 + 2 = 1$$

$$h_2 = -1 + 2 = 1$$

$$h_3 = -1 + 2 = 1$$

Hence each $h_i \geq 1$ and that is the answer.

Ex let $h_n = -h_{n-1} + 1$ $h_0 = 0$

$$h_1 = 1$$

$$h_2 = 0$$

$$h_3 = 1$$

$$h_4 = 0$$

Guess $h_n = \frac{1 + (-1)^{n+1}}{2}$

$$h_{n+1} = -h_n + 1 = -\frac{1 + (-1)^{n+1}}{2} + 1 = \frac{1}{2} - \frac{(-1)^{n+1}}{2}$$

$$= \frac{1 + (-1)^{n+2}}{2}$$

5

We will look at methods
for solving for the h_i . The
next method has a familiar
analogue in differential equations
For example:

Recall! Solve $y'' + 5y' + 6y = 0$

Let $y = e^{\delta x}$ and δ be a constant

Then $y' = \delta e^{\delta x}$, $y'' = \delta^2 e^{\delta x}$ so

$$\delta^2 e^{\delta x} + 5\delta e^{\delta x} + 6e^{\delta x} = 0 \rightarrow$$

$$\delta^2 + 5\delta + 6 = 0$$

$$\delta = 2, 3$$

$y = e^{2x}$ $y = e^{3x}$ are solutions

Since the equation is linear
and homogeneous

$$y = C_1 e^{2x} + C_2 e^{3x} \text{ gives}$$

all solutions (the general solution)

We can determine C_1 and C_2
if initial conditions are given

Suppose $y(0) = a$ $y'(0) = b$. Then
using these in the general
solution gives

7

$$C_1 + C_2 = a \quad (y(0) = a)$$

$$2C_1 + 3C_2 = b \quad (y'(0) = b)$$

Solve the system!

$$C_1 = 3a - b \quad C_2 = b - 2a$$

$$y = (3a - b)e^{2x} + (b - 2a)e^{3x} \text{ is}$$

the particular solution for these initial conditions

Linear recurrence relation solution follow a similar pattern.

Instead of $y = e^{\beta x}$ we use $h_n = \beta^n$.

Theorem. Given

$$h_n = a_1 h_{n-1} + \dots + a_k h_{n-k} \quad a_k \neq 0$$

where a_1, \dots, a_k are constants,

$h_n = \beta^n$ is a solution iff and only if β is a root of

$$x^k - a_1 x^{k-1} - \dots - a_k = 0$$

8

If the polynomial equation has distinct roots q_1, \dots, q_k

Then

$$h_n = C_1 q_1^n + \dots + C_k q_k^n \quad \text{is}$$

the general solution to the recurrence relation

Proof See p. 231 of book.

Ex Solve $h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}$

Let $h_n = x^n$

$$x^n - 2x^{n-1} - x^{n-2} + 2x^{n-3} = 0$$

Factor x^{n-3}

$$x^3 - 2x^2 + x + 2 = 0$$

$x=1$ is a root. Divide by $x-1$

Gives $(x+1)(x-2)$. The roots

$$x = 1, -1, 2 \quad \text{give}$$

$$h_n = C_1 (1)^n + C_2 (-1)^n + C_3 2^n$$

9

If we have initial conditions

$$h_0 = 1 \quad h_1 = 2 \quad h_2 = 0, \text{ use them}$$

for $n = 0, 1, 2$

$$C_1 + C_2 + C_3 = 1 = h_0$$

$$C_1 - C_2 + 2C_3 = 2 = h_1$$

$$C_1 + C_2 + 4C_3 = 0 = h_2$$

Solve these equations:

$$C_1 = 2 \quad C_2 = -\frac{2}{3} \quad C_3 = -\frac{1}{3} \quad \text{so}$$

$$h_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}2^n$$

Ex. A string of digits is generated using 0, 1, 2. No 2's can be consecutive. How many strings of n digits are possible? Let h_n stand for that number.

$$h_0 = 1 \quad h_1 = 3 \quad h_2 = 8 \text{ can be seen}$$

Suppose we are forming the strings with n digits (looking for h_n). In any $n-1$ string we can add a 0, so there are h_{n-1} strings ending with 0. Likewise for 1, so h_{n-1} strings ending with 1. The only way we can add a 2 is to end with 0 or 1. The number of strings ending with 0 ($n-1$ digits) is h_{n-2} since we can add a 0 to end $n-2$ strings. Likewise for $n-1$ digits ending with 1. So ~~for~~ to add 2 there are h_{n-2} cases

$$\text{Total } 2h_{n-1} + 2h_{n-2} = h_n \quad \text{So}$$

$$h_n - 2h_{n-1} - 2h_{n-2} = 0 \quad \text{or}$$

$$x^n - 2x^{n-1} - 2x^{n-2} = 0$$

$$x^2 - 2x - 2 = 0$$

$$x = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$$

11

The general solution

$$h_n = C_1 (1 + \sqrt{3})^n + C_2 (1 - \sqrt{3})^n$$

To find C_1, C_2 use $h_0 = 1, h_1 = 3$

$$1 = h_0 = C_1 + C_2$$

$$3 = h_1 = C_1 (1 + \sqrt{3}) + C_2 (1 - \sqrt{3})$$

Solve $C_1 = \frac{2 + \sqrt{3}}{2\sqrt{3}}$

$$C_2 = \frac{-2 + \sqrt{3}}{2\sqrt{3}}$$

$$h_n = \frac{2 + \sqrt{3}}{2\sqrt{3}} (1 + \sqrt{3})^n + \frac{-2 + \sqrt{3}}{2\sqrt{3}} (1 - \sqrt{3})^n$$

Problems Page 261-262

33, 38, 34, 35

Example $h_n = 4h_{n-1} - 4h_{n-2}$

$$h_n - 4h_{n-1} + 4h_{n-2} = 0$$

$$h_n = x^n$$

$$x^n - 4x^{n-1} + 4x^{n-2} = 0$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)(x-2) = 0$$

The roots are repeated

We can not use

$$h_n = C_1 2^n + C_2 2^n$$

as this only gives one solution

and we need two (since

$x^2 - 4x + 4$ is a quadratic)

Notice that $h_n = C_1 2^n + C_2 n 2^n$
is a solution. (We need only verify
 $n 2^n$ is a solution)

$$n 2^n - 4(n-1) 2^{n-1} + 4(n-2) 2^{n-2}$$

$$= 2^{n-2} [n 2^2 - 4(n-1) 2 + 4(n-2)]$$

$$= 2^{n-2} [4n - 8n + 8 + 4n - 8] = 0$$

So the general solution is

$$h_n = C_1 2^n + C_2 n 2^n$$

The idea generalizes: If a is repeated as a root k times, use

$$h_n = C_1 a^n + C_2 n a^n + \dots + C_k (n-1) a^n$$

Problems Ch 7 33, 34, 35, 38

Ex. h_n denotes the number of ways to color a $4 \times n$ board with colors red, white, blue and green with the

number of red squares is even

number of white is odd

Find generating function

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \left(1 + x + \frac{x^2}{2!} + \dots\right) \left(1 + x + \frac{x^2}{2!} + \dots\right)$$

$$= \frac{e^x + e^{-x}}{2} \frac{e^x - e^{-x}}{2} e^x e^x = \frac{1}{4} [e^{4x} - 1]$$

$$= \frac{1}{4} \left[1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \frac{(4x)^n}{n!} + \dots \right] - \frac{1}{4}$$

$$h_0 = 0 \quad h_n = \frac{4^n}{4} = 4^{n-1}$$