

Ma 416

Lesson 25

Generating Function Solutions to Linear Recurrence Relations

Ex. Solve $h_n = 5h_{n-1} - 6h_{n-2}$

$$\text{with } h_0 = 1 \quad h_1 = -2$$

$$h_n - 5h_{n-1} + 6h_{n-2} = 0$$

$$\text{Let } g(x) = \sum_{n=0}^{\infty} h_n x^n$$

We will plug $g(x)$ into the linear recurrence relation. So

$$\begin{aligned} g(x) &= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \dots \\ -5xg(x) &= -5h_0 x - 5h_1 x^2 - 5h_2 x^3 + \dots \\ 6x^2g(x) &= 6h_0 x^2 + 6h_1 x^3 + \dots \end{aligned}$$

Add these together

$$\begin{aligned} g(x) - 5xg(x) + 6x^2g(x) &= h_0 + (h_1 - 5h_0)x \\ &\quad + (h_2 - 5h_1 + 6h_0)x^2 + (h_3 - 5h_2 + 6h_1)x^3 \end{aligned}$$

All terms from x^2 are 0 because

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of the recurrence relation. So

$$(1 - 5x + 6x^2) g(x) = h_0 + (h_1 - 5h_0)x$$

$$\text{Since } h_0 = 1 \quad h_1 = -2$$

$$(1 - 5x + 6x^2) g(x) = 1 - 7x \rightarrow$$

$$g(x) = \frac{1 - 7x}{1 - 5x + 6x^2}$$

We use partial fractions

$$\frac{1 - 7x}{1 - 5x + 6x^2} = \frac{A}{1-2x} + \frac{B}{1-3x}$$

$$\text{Since } 1 - 5x + 6x^2 = (1-2x)(1-3x)$$

OR

$$1 - 7x = A(1-3x) + B(1-2x)$$

so

$$1 = A + B$$

$$-7 = -3A - 2B$$

Solve this system to get

$$A = 5 \quad B = -4$$

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Use the geometric series to get

$$\frac{1}{1-2x} = 1 + (2x)^1 + (2x)^2 + (2x)^3 \\ = 1 + 2x + 2^2 x^2 + 2^3 x^3 \dots$$

$$\frac{1}{1-3x} = 1 + 3x + (3x)^2 \dots \\ = 1 + 3x + 3^2 x^2 + 3^3 x^3 \dots \rightarrow$$

$$g(x) = 5(1 + 2x + 2^2 x^2 + \dots) \\ - 4(1 + 3x + 3^2 x^2 + \dots) \\ = 1 - 2x - 16x^2 \dots \\ + (5 \cdot 2^n - 4 \cdot 3^n)x^n \dots$$

$$\text{So } h_n = 5 \cdot 2^n - 4 \cdot 3^n$$

4. Ex $h_0, \dots, h_n \dots$ is a sequence which satisfies

$$h_n + h_{n-1} - 6h_{n-2} + 20h_{n-3} = 0$$

with initial conditions

$$h_0 = 0 \quad h_1 = 1 \quad h_2 = -1$$

Proceed as in the last example

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

$$g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 + \dots$$

$$x g(x) = h_0 x + h_1 x^2 + h_2 x^3 + h_3 x^4 + \dots$$

$$-16x g(x) = -16h_0 x - 16h_1 x^2 - 16h_2 x^3 - 16h_3 x^4 + \dots$$

$$+ 20x^3 g(x) = 20h_0 x^3 + 20h_1 x^4 + \dots$$

Add

$$(1 + x - 16x^2 + 20x^3) g(x) = h_0 + (h_1 + h_0)x$$

$$+ (h_2 + h_1 - 16h_0)x^2 + \underbrace{(h_3 + h_2 - 16h_1 + 20h_0)}_0 x^3$$

$$+ 0$$

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Put in the initial conditions
to get

$$(1+x-16x^2+20x^3)g(x) = x$$

$$g(x) = \frac{x}{1+x-16x^2+20x^3}$$

Factor the denominator!

$$1+x-16x^2+20x^3 = (1-2x)^2(1+5x) \rightarrow$$

$$\frac{x}{1+x-16x^2+20x^3} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+5x} \rightarrow$$

$$x = A(1-2x)(1+5x) + B(1+5x) + C(1-2x)^2$$

$$\text{when } x = 1/2$$

$$\frac{1}{2} = 0 + \frac{1}{2}B + 0 \rightarrow B = 1/7$$

$$\text{when } x = -1/5$$

$$-\frac{1}{5} = C(1+2/5)^2 = C\left(\frac{49}{25}\right) \rightarrow C = -\frac{5}{49}$$

$$\text{when } x = 0$$

$$0 = A + B + C = A + 1/7 - 5/49 \rightarrow$$

$$A = -2/49$$

$$6 \quad g(x) = \frac{2/49}{1-2x} + \frac{1/7}{(1-2x)^2} - \frac{5/49}{1+5x}$$

Now

$$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + \dots + 2^n x^n$$

$$\frac{1}{1+5x} = 1 - 5x + 5^2x^2 - \dots (-1)^n 5^n x^n$$

$$\frac{1}{(1-2x)^2} = 1 + \binom{2}{1}x + \binom{3}{2}2^2x^2 + \dots \binom{n+1}{n}2^n x^n$$

$$g(x) = -\frac{2}{49} \left\{ 2^n x^n + \frac{1}{7} \sum (n+1) 2^n x^n \right\} + \frac{5}{49} \left\{ (-5)^n x^n \right\}$$

$$\rightarrow h_n = -\frac{2}{49} 2^n + \frac{1}{7} (n+1) 2^n - \frac{5}{49} (-5)^n$$

Homework

Ch 7 48 (a c d e)

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Ex Solve

$$h_n = 8h_{n-1} - 16h_{n-2}$$

$$h_0 = -1 \quad h_1 = 0$$

$$h_n - 8h_{n-1} + 16h_{n-2} = 0$$

$$\text{Let } h_n = x^n$$

$$x^n - 8x^{n-1} + 16x^{n-2} = 0$$

$$x^{n-2} [x^2 - 8x + 16] = 0$$

$$x^2 - 8x + 16 = 0$$

$$(x-4)^2$$

$$h_n = C_1 4^n + C_2 n 4^n$$

Use initial conditions

$$-1 = C_1$$

$$0 = 4C_1 + 4C_2 \rightarrow C_2 = 1$$

$$h_n = -4^n + n 4^n$$

Ex Solve the recurrence by

Looking at the first few values

to conjecture a solution. Then show it works

$$h_n = 2h_{n+1} + 1 \quad h_0 = 1$$

$$h_0 = 1$$

$$h_1 = 3$$

$$h_2 = 7$$

$$\text{Guess } h_n = 2^n - 1$$

$$h_3 = 15$$

$$h_4 = 31$$

$$h_{n+1} = 2h_n + 1 = 2(2^n - 1) + 1 \\ = 2^{n+1} - 1$$

So result holds.