MA 411

Lesson 26

Solutions to nonhomogenious linear recurrance relations with constant coefficients.

We now look at linear recurrance relations where the right hand side is not 0. There are several methods The first is like solutions of differential equations that are not homogenious. We first solve the homogenious case and the a particular case for the original recurrance and add the answers. In this method we focus on two possible right hand sides

1. A polynomial of degree s
2. a^n

In both cases we make a guess at the general form. The first case the guess is a polynomial of degree s. The second is a n exponential

The second method is by generating functions. First, the first method

Ex $h_n = 2h_{n-1} + 1$ $h_0 = 0$

Write this as $h_n - 2h_{n-1} = 1$

The homogenious case is $h_n - 2h_{n-1} = 0$

let $h_n = x^n$. Get $x^n - 2x^{n-1} = 0$. So

$x^{n-1}(x-2) = 0 \rightarrow x = 2$ The homogeneous

solution is $h_n = c2^n$

In $h_n - 2h_{n-1} = 1$, we guess at a

polynomial of degree 0: $h_n = r$

$\rightarrow$ Substitute into the equation:

$$r = +2r + 1$$

$$r = -1$$

Adding:

So $h_n = c2^n - 1$

The initial condition $h_0 = 0 \rightarrow$

$0 = c - 1 \rightarrow c = 1$

$$h_n = 2^n - 1.$$

Ex $h_n = 3h_{n-1} - 4n$

Look at $h_n - 3h_{n-1} = 0 \rightarrow h_n = x^n$

$$x^n - 3x^{n-1} = 0$$

$$x^{n-1}(x-3) = 0 \qquad x = 3$$

$h_n = c3^n$ is the homogeneous

solution

3 For the particular solution, since the right hand side is $-4n$, let

$h_n = rn + s$ (find r and s)

$\rightarrow \quad rn + s = 3[r(n-1)+s] - 4n$

$r = 3r - 4$ (coefficient of n)

$s = -3r + 3s$

So $2r = 4 \rightarrow r = 2 \rightarrow s = -6 + 3s$ or $2s = 6$, $s = 3$. So $h_n = 2n + 3$ is the particular solution and

$h_n = c3^n + 2n + 3$ is the general solution

If we are given $h_0 = 2$, then

$2 = c + 3 \rightarrow c = -1$

$h_n = -3^n + 2n + 3$

Ex $h_n = 2h_{n-1} + 3^n \qquad h_0 = 2$

Look at $h_n = 2h_{n-1}$ or $h_n - 2h_{n-1} = 0$

$h_n = x^n \qquad x^n - 2x^{n-1} = 0 \rightarrow x^{n-1}(x-2)$

So $h_n = c2^n$

The right hand side of the original equation is 3^n. For the particular solution, try $h_n = d\,3^n$

Substitute into original

$$d\,3^n = 2d\,3^{n-1} + 3^n$$

$$\rightarrow \quad 3d = 2d + 3$$

$$\rightarrow \quad d = 3$$

The particular solution is

$$h_n = d\,3^n = 3^{n+1}$$

Adding

$$h_n = c\,2^n + 3^{n+1} \qquad \text{Since } h_0 = 2$$

$$2 = c + 3 \qquad c = -1$$

Finally $h_n = -2^n + 3^{n+1}$

Ex. $h_n = 3h_{n-1} + 3^n$, $h_0 = 2$

Like ~~As~~ in the last example, the homogeneous solution is

$$h_n = c\,3^n$$

Try $h_n = p\,3^n$ for the particular solution. From $h_n = 3h_{-1} + 3^n$ we get

$$p\,3^n = 3p\,3^{n-1} + 3^n$$

Cancel 3^{n-1}: $\quad p\,3 = 3p + 3$

which has no solution. Try

$$h_n = n\,p\,3^n$$

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Substitute to get

$$p n 3^n = 3p(n-1)3^{n-1} + 3^n$$

$$3pn = 3p(n-1) + 3$$

$$p = 1$$

$\rightarrow$ $h_n = n 3^n$ is the particular solution

The solution is gotten by adding

$$h_n = c 3^n + n 3^n$$

To find c, use $h_0 = 2$

$$2 = c$$

$\rightarrow$ $h_n = 2 \cdot 3^n + n 3^n$ is the solution

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Recurrance Relation solutions by Generating Functions

Ex. $h_n = 2h_{n-1} + 3^n \qquad h_0 = 2$

Let $g(x) = h_0 + h_1 x + \cdots + h_n x^n + \cdots$

$2xg(x) = \qquad 2h_0 x \qquad 2h_{n-1}x^n \cdots$

Subtract

$g(x) - 2xg(x) = h_0 + (h_1 - 2h_0)x + (h_2 - 2h_1)x \cdots$

$(h_n - 2h_{n-1})x^n$

But $h_n - 2h_{n-1} = 3^n$ and $h_0 = 2$. $\rightarrow$

$g(x) - 2xg(x) = 2 + 3x + 3^2x^2 + \cdots + 3^n x^n \cdots$

$= 2 - 1 + 1 + 3x + 3^2x^2 + \qquad + 3^n x^n$

$= 1 + \dfrac{1}{1-3x}$

$g(x) = \dfrac{1}{1-2x} + \dfrac{1}{(1-3x)(1-2x)}$

$\dfrac{1}{(1-3x)(1-2x)} = \dfrac{A}{1-3x} + \dfrac{B}{1-2x}$

$1 = A(1-2x) + B(1-3x)$

$x = 1/2$: $\quad 1 = 0 + B(-1/2) \qquad B = -2$

$x = \frac{1}{3} \quad 1 = A(1/3) \qquad A = 3$

$g(x) = \dfrac{1}{1-2x} + \dfrac{3}{1-3x} - 2\dfrac{1}{1-2x}$

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$$S(x) = (1 + 2x + 2^2x^2 + \dots)$$

$$+ 3(1 + 3x + 3^2x^2 + \dots)$$

$$- 2(1 + 2x + 2^2x^2 + \dots)$$

$$= \sum 2^n x^n + 3\sum 3^n x^n - 2\sum 2^n x^n$$

$$= \sum (2^n + 3^{n+1} - 2^{n+1}) x^n$$

$$h_n = 2^n + 3^{n+1} - 2^{n+1}$$

Ex Solve $h_n = 3h_{n-1} + 3^n \qquad h_0 = 2$

Let $\cancel{h_n = 3^n}$

$$g(x) = h_0 + h_1 x + \qquad + h_n x^n$$

$$-3x\,g(x) \qquad -3h_0 x \qquad -3h_{n-1}x^n$$

Add

$$g(x) - 3x\,g(x) = h_0 + (h_1 - 3h_0)x \cdots + (h_n - 3h_{n-1}x^n)$$

$$= h_0 + 3x + 3^2x^2 \qquad + 3^n x^n$$

$$= 2 + 3x + 3^2x^2 + \qquad + 3^n x^n$$

$$= 2 - 1 + 1 + 3x + \qquad + 3^n x^n$$

$$= 1 + \frac{1}{1-3x} \rightarrow$$

$$g(x) = \frac{1}{1-3x} + \frac{1}{(1-3x)^2}$$

$$= \sum 3^n x^n + \binom{n+1}{n} 3^n x^n$$

$$= \sum 3^n + \binom{n+1}{n} 3^n \; x^n$$

$$h_n = 3^n + (n+1)\,3^n$$

$$= (n+2)\,3^n$$

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