

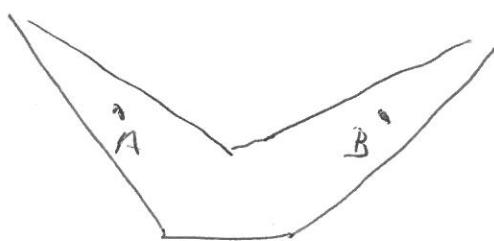
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Lesson 27

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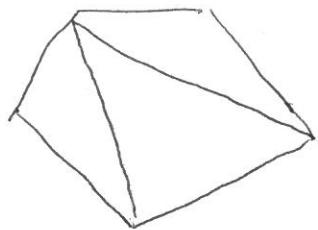
Today we will have a more complicated recurrence. It comes from a triangulation problem. The thing that arises is classic. They are Catalan numbers, occur incredibly often, and we will look at them in the next chapter.

Consider a region in the plane that is bounded by  $n$  line segments, so has  $n$  vertices. A triangle, square, pentagon are all examples. So is a trapezoid. The region is convex if for every pair of points in the region, the line segment joining them is entirely in the region. An example of a non-convex region!

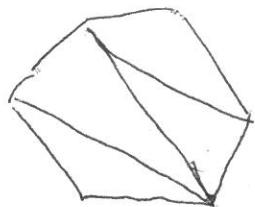


A and B fail the convex test.

2 The convex region is said to be triangulated if we can draw segments connecting vertices such that the resulting regions are all triangles. The number of vertices



In the region is  $n=5$  and the number of diagonals is  $n-3 = 2$



Here  $n=6$  and the number of diagonals is  $n-3 = 3$

We are going to triangulate with diagonals that do not intersect (except at vertices)

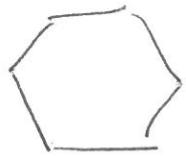
Before we put that restriction on the problem, we ask in such a figure with  $n$  vertices, how many

3) How many diagonals are there? (For now, we are allowed to intersect). For each vertex, there are  $n-3$  other vertices that connect to the vertex with a diagonal. There are  $n$  vertices, giving  $n(n-3)$  diagonals. Each has been counted twice, so the total number of diagonals is

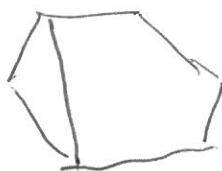
$$\frac{n(n-3)}{2}$$

Remember, we are not counting the sides of the figure.

Now, since the region is convex, each diagonal divides the region into two convex regions, one with  $k$  sides and one with  $n-k+2$  sides.



$$n=6$$



The left figure has 3 sides

The right figure has

$$S = n - k + 2 = 6 - 3 + 2 \text{ sides}$$

This is because, the diagonal has added

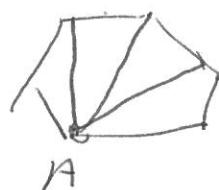
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a side to each of the parts.

We can triangulate by picking one vertex and drawing a segment to each of the  $n-3$  other vertices which are not adjacent by a side and not the vertex



$$n = 6$$



$$n-3 = 3$$

$$\begin{aligned} n &= 6 \\ \text{regions} &= n-2 = 4 \\ \text{diagonals} &= n-3 = 3 \end{aligned}$$

This triangulates the region and can always work. The question we ask is how many ways can we triangulate a convex region which has  $n$  vertices and the triangulation does not allow diagonals to intersect. Given  $n$  vertices, there will be  $n-3$  diagonals and  $n-2$  regions, as in the above figure.

5-

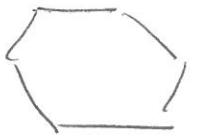
The derivation works better if we suppose there are  $n+1$  vertices (so in the last figure,  $n=5$ )

There are then  $n-2$  diagonals and  $n-3$  triangular regions

Thm. Let  $h_n$  be the number of ways of dividing a convex polygon with  $n+1$  vertices ( $= n+1$  sides) into triangular regions by inserting diagonals that do not intersect (except at vertices). Define  $h_1 = 1$ . Then  $h_n$  satisfies the recurrence relation

$$h_n = h_1 h_{n-1} + h_2 h_{n-2} + \dots + h_{n-2} h_2 + h_{n-1} h_1.$$

The solution is  $h_n = \frac{1}{n} \binom{2n-2}{n-1}$



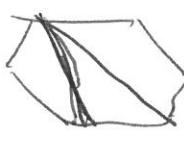
BASE

Pick one side and call it the base

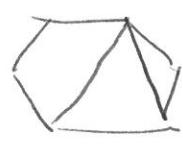
We go around the polygon getting  
the elements in the sum



1



2



3



4

In all but the end cases, we see  
the middle triangle breaks the region  
into 2 other convex polygons

The left polygon has ~~k+1 sides~~ sides

The right polygon has ~~n-k+1~~ sides

$$\text{In Fig 2 } k=2 \quad n-k+1=5-2+1=4$$

$$\text{In Fig 3 } k=3 \quad n-k+1=5-3+1=3$$

Call the left region A, the right B

A has  $k+1$  sides B has  $n-k+1$  sides

So A can be triangulated in  $h_k$  ways

B can be triangulated in  $h_{n-k}$  ways

So there are  $h_k h_{n-k}$  ways to  
triangulate both for this  $k$ .

We also consider the end pictures. Say,  
There  $h_1=1$  is the lone on the left

7 So including B, we get  $h_1 h_{n-1}$  ways to triangulate them. For the last region the same reasoning gives  $h_{n-1} h_1$  ways.

So as the triangles  involving

the base and each vertex gives the total

$$h_1 h_{n-1} + h_2 h_{n-2} + \dots + h_{n-1} h_1 = h_n$$

Computations

$$h_1 = 1$$

$$h_2 = h_1 h_1 = 1$$

$$h_3 = h_1 h_2 + h_2 h_1 = 1 + 1 = 2$$

$$h_4 = h_1 h_3 + h_2 h_2 + h_3 h_1 = 2 + 1 + 2 = 5$$

$$h_5 = h_1 h_4 + h_2 h_3 + h_3 h_2 + h_4 h_1 = 5 + 2 + 2 + 5 = 12$$

To solve the recurrence, we use generating functions

$$\begin{aligned}
 g(x) &= h_1 x + h_2 x^2 + h_3 x^3 + \dots \\
 (g(x))^2 &= h_1^2 x^2 + (h_1 h_2 + h_2 h_1) x^3 + \\
 &\quad (h_1 h_3 + h_2 h_2 + h_3 h_1) x^4 + \dots \\
 &\quad (h_1 h_{n-1} + \dots + h_{n-1} h_1) x^n \\
 &= h_1^2 x^2 + h_3 x^3 + h_4 x^4 + \dots + h_n x^n \\
 &= g(x) - h_1 x = g(x) - x \quad (h_1^2 = h_1 h_1 = h_2)
 \end{aligned}$$

In the last line we added and subtracted  $h_1 x$  from the preceding line and also used  $h_1 = 1$

$$So \quad g(x)^2 - g(x) + x = 0$$

The quadratic in  $g(x)$  gives

$$g(x) = \frac{1 + \sqrt{1 - 4x}}{2}, \quad \frac{1 - \sqrt{1 - 4x}}{2}$$

Call these  $g_1(x)$  and  $g_2(x)$

$$Now \quad g(0) = 0, \quad g_1(0) = 1, \quad g_2(0) = 0 \rightarrow$$

$$g(x) = g_2(x) = \frac{1 - \sqrt{1 - 4x}}{2} = \frac{1}{2} - \frac{1}{2} (1 - 4x)^{\frac{1}{2}}$$

9

Newton's Binomial Theorem gives

$$(1+x)^{1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^{2n-1}} \binom{2n-2}{n-1} x^n$$

Replace  $x$  by  $-4x$

$$\begin{aligned} (1-4x)^{1/2} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^{2n-1}} \binom{2n-2}{n-1} (-1)^n 4^n x^n \\ &= 1 + \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{2}{n} \binom{2n-2}{n-1} x^n \\ &= 1 - 2 \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n \end{aligned}$$

$$\therefore g(x) = \frac{1}{2} - \frac{1}{2} (1-4x)^{1/2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n$$

$$\rightarrow h_n = \frac{1}{n} \binom{2n-2}{n-1}$$

Again, the  $h_n$  are called  
the Catalan numbers