

M A 416

Lesson 28

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Catalan Numbers

Last time we introduced the Catalan numbers as a solution to a geometric problem: let R be a convex set whose boundary consists of $n+1$ line segments. Hence R has $n+1$ vertices. The problem is to find the number of ways, h_n , to triangulate R with non intersecting diagonals whose end points are vertices.

We set $h_1 = 1$

$$n=2 \quad \triangle \quad h_2 = 1$$

$$n=3 \quad \square \quad h_3 = 2$$

We found that

$$h_n = h_1 h_{n-1} + h_2 h_{n-2} + \dots + h_{n-1} h_1$$

$$\text{Thus } h_4 = h_1 h_3 + h_2 h_2 + h_3 h_1 = 2 + 1 + 2 = 5$$

$$h_5 = h_1 h_4 + h_2 h_3 + h_3 h_2 + h_4 h_1 = \\ 5 + 2 + 2 + 5 = 14$$

2 The Catalan numbers are solutions to a huge number of problems. We will look at some of them.

We saw that if R has $n+1$ sides, then $h_n = \frac{1}{n+1} \binom{2n}{n}$. When $n=3$, the 4 sided square can be triangulated in $\frac{1}{3} \binom{4}{2} = 2$ ways.

Define $C_n = \frac{1}{n+1} \binom{2n}{n}$. These are the Catalan numbers. So

$C_2 = h_3 = 2$

$$C_3 = h_4 = 5$$

$C_{n-1} = \frac{1}{n} \binom{2n-2}{n-1} = h_n$ is the number of ways of triangulating an $n+1$ figure.

Ex square ~~not~~ $n+1=4$ $n=3$

$$C_2 = \frac{1}{3} \binom{4}{2} = 2$$

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The first few values

$$C_0 = 1$$

$$C_1 = 1$$

$$C_2 = h_3 = 2$$

$$C_3 = h_4 = 5$$

$$C_4 = h_5 = 14$$

$$C_5 = h_6 = 42$$

$$C_6 = h_7 = 132$$

Problem. The number of sequences

a_1, \dots, a_{2n} with n 1's and n -1's

whose partial sum is non negative

equals $C_n = \frac{1}{n+1} \binom{2n}{n}$

Proof. As in the book, call a sequence of n 1's and n -1's acceptable if it satisfies the condition, unacceptable otherwise

Let A_n be the number of acceptable sequences and U_n be the number of unacceptable sequences. So

$$A_n + U_n = \text{total number} = \binom{2n}{n} = \frac{(2n)!}{n! n!}$$

Since it is counting a multiset permutation problem with $n!$'s and $n-1$'s.

We find U_n . If a sequence is in U_n , there is a first place where $a_1 + \dots + a_k < 0$. Then $a_1 + \dots + a_{k-1} = 0$, $k-1$ is even and k is odd. Form a new sequence by ~~repl~~ change pos to neg and neg to pos up to k . Leave the rest unchanged. The old sequence had one more -1 up to the k term so the new sequence has $n+1$ 1's and $n-1$ -1 's.

Conversely, a sequence of $n+1$ 1's and $n-1$ -1 's, there is a first place the number of 1's is more than the number of -1 's. Change signs in every term up to that point and get an unacceptable sequence. So the number of unacceptable sequences

\therefore The number of sequences with $n+1$ 1's and $n-1$ -1's, The T number is

$$\binom{2n}{n+1} = \frac{(2n)!}{(n+1)!(n-1)!}$$
 (Again a multiset)

Permutation problem with $n+1$ 1's
 $n-1$ -1's. So

$$U_n = \frac{(2n)!}{(n+1)!(n-1)!}$$

Therefore

$$A_n = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!}$$

$$= \frac{(2n)!}{n!(n-1)!} \left[\frac{1}{n} - \frac{1}{n+1} \right]$$

$$= \frac{(2n)!}{n!(n-1)!} \frac{1}{n(n+1)}$$

$$= \frac{2n!}{n!n!} \frac{1}{n+1} = \frac{1}{n+1} \binom{2n}{n} = C_n$$

Many problems that have a different look can be viewed as an example of last problem

For example:

Point B is 10 blocks north and 10 blocks east of point A. How many paths are there from A to B if we must stay above or on a diagonal line from A to B



can't be here

In any path we take we can not be more blocks east than north at any time. Replace a block north with a 1 and a block east with a -1. The paths from A to B are a sequence which contains 10 1's and 10 -1's. After each block form the partial sums accrued on the trip. The sum must always be ≥ 0 . This is

the last problem and the answer is

$$C_{10} = \frac{1}{10+1} \binom{20}{10} = \frac{1}{11} \binom{20}{10}$$

If, instead of insisting on being
above or on the diagonal, we just
insist that it is not crossed,
then we have the same number of
paths ~~on~~ on the other side, so we
get a total of

$$\frac{2}{11} \binom{20}{10}$$

Example 2n people are in line
to get into a movie that
cost's 50¢. n of the people
have 50¢ in change and 50
of the people have a dollar bill.
The movie theater starts with
no change. How many ways
can the people line up if

a. It only matters if there is change to give out

b. The exact order of the people also matters

a. Clearly we need a person with change to start. After that we need to have enough change accumulated to give it out. There must always be more ^(or equal) people with change going in line than those with bills.

Give a 1 to those with change

and -1 to those with dollar bills.

Any partial sum must have more than or equal to change than dollars. So the partial sums must

be ≥ 0 . This is the same problem again, so the answer is

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

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b If the order of the people matter,
those with change can be arranged
in $n!$ ways, as can those with bills.
Thus the answer is

$$n! n! \frac{1}{n+1} \binom{2n}{n}$$

10 Catalan numbers satisfy the a
a homogeneous recurrence with
non constant coefficients. To find it

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} \frac{(2n)!}{n! n!}$$

$$C_{n-1} = \frac{1}{n} \binom{2(n-1)}{n-1} = \frac{1}{n} \frac{(2n-2)!}{(n-1)! (n-1)!}$$

Then $\frac{C_n}{C_{n-1}} = \frac{n}{n+1} \frac{2n(2n-1)}{n n} = \frac{4n-2}{n+1}$

$$\Rightarrow C_n = \frac{4n-2}{n+1} C_{n-1} \quad \text{with } C_0=1$$

Problems Ch. 8 1, 2